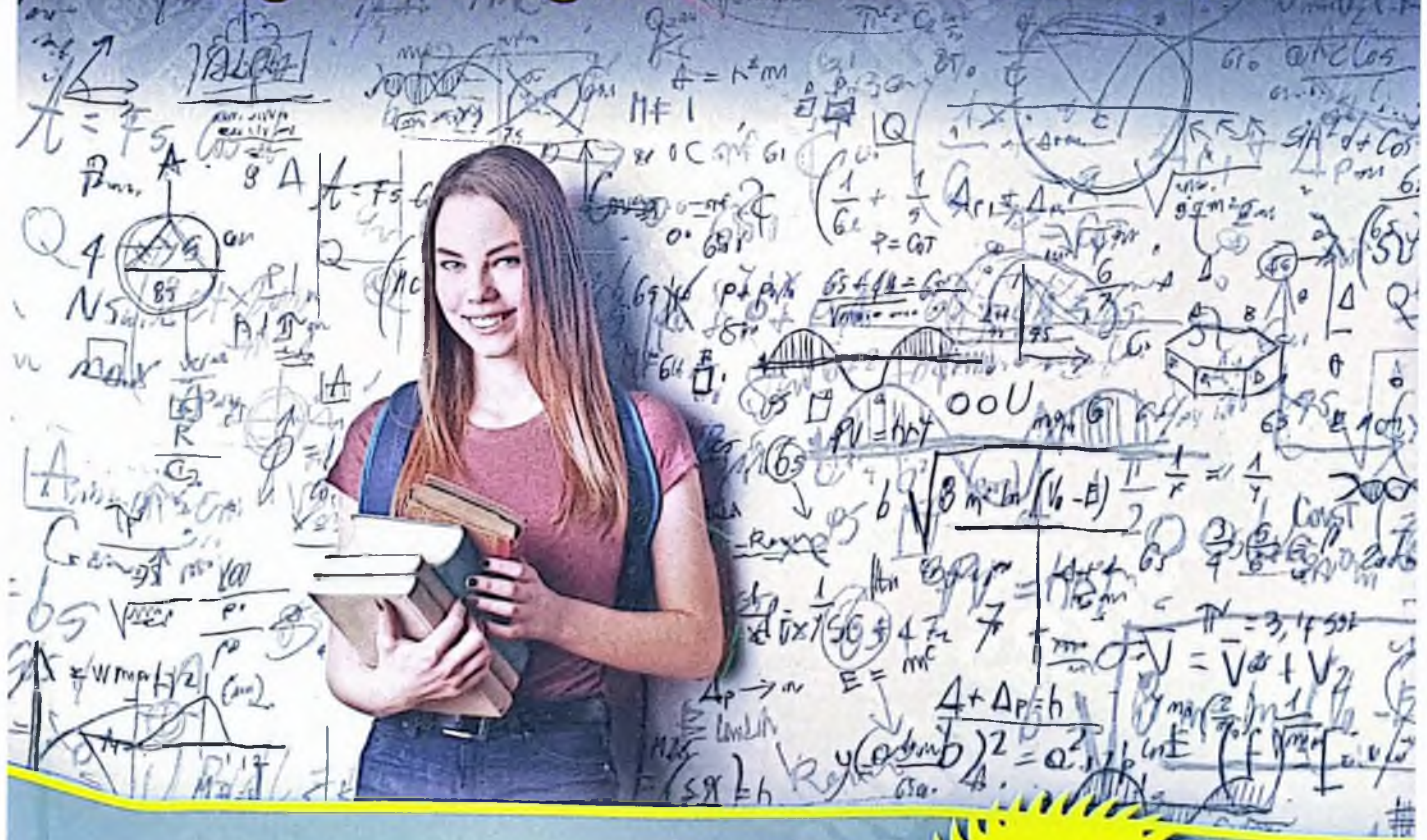


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Engineering Mathematics - II

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CONTENTS**BAS203 : Engineering Mathematics - II****UNIT-1 : ORDINARY DIFFERENTIAL EQUATION OF HIGHER ORDER**

(1-1 F to 1-36 F)

Linear differential equation of n^{th} order with constant coefficients, Simultaneous linear differential equations, Second order linear differential equations with variable coefficients, Solution by changing independent variable, Method of variation of parameters, Cauchy-Euler equation, Application of differential equations in solving engineering problems.

UNIT-2 : LAPLACE TRANSFORM

(2-1 F to 2-27 F)

Laplace transform, Existence theorem, Properties of Laplace Transform, Laplace transform of derivatives and integrals, Unit step function, Laplace transform of periodic function, Inverse Laplace transform, Convolution theorem, Application of Laplace Transform to solve ordinary differential equations and simultaneous differential equations.

UNIT-3 : SEQUENCE AND SERIES

(3-1 F to 3-29 F)

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UNIT-4 : COMPLEX VARIABLE-DIFFERENTIATION

(4-1 F to 4-33 F)

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1 UNIT

Ordinary Differential Equation of Higher Order

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of n^{th} Order with
Constant Coefficients
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1-2 F (Sem-2)

Ordinary Differential Equation of Higher Order

PART-1

*Linear Differential Equations of n^{th} Order
with Constant Coefficients.*

CONCEPT OUTLINE

Differential Equation : An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

For example, $\log\left(\frac{dy}{dx}\right) = ax + by$

$$(1 - x^2)(1 - y)dx = xy(1 + y)dy$$

$$\frac{dy}{dx} = \sec(x + y)$$

Order of a Differential Equation : The order of a differential equation is the order of the highest derivative involved in a differential equation.

For example, $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^5$ is of 4th order.

Degree of a Differential Equation : The degree of a differential equation is the power of the highest derivative which occurs in it, after the differential equation has been made free from radicals and fractions as far as the derivatives are concerned.

For example, $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^4$, is of first degree.

Linear Differential Equation : A linear differential equation is an equation in which the dependent variable and its derivatives appear only in the first degree.

For example, $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 9y = 4x^2 - 7$

The above equation is called a LDE (linear differential equation) with constant coefficients.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.1. Write the procedure to find complementary function.

Answer

Following are the steps to find complementary function :

Step I : Put the RHS of the given equation equals to zero. i.e., $f(D)y = 0$

Step II : Replace $\frac{d}{dx} \approx D$, $\frac{d^2}{dx^2} \approx D^2$ and so on i.e., convert the given equation in symbolic form.

Step III : Make an auxiliary equation replacing D by m .

e.g., $(D^2 + 4D + 7) = 0$ then its auxiliary equation is $m^2 + 4m + 7 = 0$

Step IV : Find the roots of auxiliary equation, complementary function (CF) will depend upon the type of root.

Case I : If all roots of the auxiliary equation are real and distinct say $m_1, m_2, \dots, m_n$

Then, $CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$

Where, $C_1, C_2, \dots, C_n$ are constants.

Case II : If roots of auxiliary equation are real and equal say

$$m_1 = m_2 = \dots = m_n = m \text{ (say).}$$

Then, $CF = (C_1 + C_2 x + C_3 x^2 + \dots + C_n x^{n-1}) e^{mx}$

If some roots are equal, others are distinct say

$$m_1 = m_2 = m_3 = m$$

and $m_4, m_5, \dots, m_n$

Then, $CF = (C_1 + C_2 x + C_3 x^2) e^{mx} + C_4 e^{m_4 x} + C_5 e^{m_5 x} + \dots + C_n e^{m_n x}$

Case III : If the roots of auxiliary equation are complex say

$$m = \alpha \pm i\beta, \text{ then}$$

$$CF = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

$$\text{or } CF = C_1 e^{\alpha x} e^{i\beta x} + C_2 e^{\alpha x} e^{-i\beta x}$$

$$CF = C_1 e^{\alpha x} (\cos \beta x + i \sin \beta x) + C_2 e^{\alpha x} (\cos \beta x - i \sin \beta x)$$

$$CF = e^{\alpha x} (C_1 + C_2) \cos \beta x + i e^{\alpha x} (C_1 - C_2) \sin \beta x$$

$$CF = e^{\alpha x} [A \cos \beta x + iB \sin \beta x]$$

or changing the constants

$$CF = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

This expression may be written as

$$CF = C_1 e^{\alpha x} (\cos \beta x + C_2)$$

$$\text{or } CF = C_1 e^{\alpha x} (\sin \beta x + C_2)$$

Case IV : If the auxiliary equation has irrational roots say

$$m = \alpha \pm \sqrt{\beta}, \text{ where } \beta \text{ is positive}$$

$$\text{Then, } CF = e^{\alpha x} (C_1 \cosh \sqrt{\beta} x + C_2 \sinh \sqrt{\beta} x)$$

Que 1.2. Explain the method to find out the particular integral when the function in RHS is e^{ax} , $f(a) \neq 0$ and e^{ax} , $f(a) = 0$.

Answer

Case I :

When RHS function is e^{ax} , $f(a) \neq 0$,

$$\text{Then, } PI = \frac{1}{f(D)} e^{ax}$$

Now replace D by a so PI will be,

$$= \frac{e^{ax}}{f(a)}$$

If $f(a) = 0$, it will be a case of failure.

Case II :

When RHS of function is e^{ax} , $f(a) = 0$,

$$\text{Then, } PI = \frac{e^{ax}}{f(D)}$$

$$\text{Now, } PI = \frac{x e^{ax}}{f'(D)}$$

Multiply with x and differentiate denominator once.

Again if $f''(a) = 0$ then, continue to multiply with x and differentiate denominator,

$$PI = x^n \frac{e^{ax}}{f^n(a)}$$

Que 1.3. What is the procedure to find particular integral when the RHS function is either $\sin ax$, $\cos ax$ while $f(-a^2) \neq 0$, or $\sin ax$, $\cos ax$ while $f(-a^2) = 0$?

Answer

Case I : When function is $\sin ax$ or $\cos ax$ and $f(-a^2) \neq 0$,

$$PI = \frac{\sin ax}{f(D^2)}$$

or $PI = \frac{\cos ax}{f(D^2)}$

In both cases replace D^2 by $-a^2$ but $f(-a)^2 \neq 0$. If after replacing D^2 by $-a^2$ any term of D exist in denominator then, multiply the operator by its conjugate, again D^2 by $-a^2$. Terms of D in numerator stands for differentiation of function.

Case II : When function is $\sin ax$ or $\cos ax$ and $f(-a^2) = 0$,

$$PI = \frac{\sin ax}{f(D^2)} = x \frac{\sin ax}{f'(-a^2)}$$

Repeat this step again if $f'(-a^2) = 0$.

Que 1.4. Solve $\frac{d^2 y}{dx^2} + 4y = \sin^2 2x$ with conditions $y(0) = 0$,

$y'(0) = 0$.

AKTU 2012-13, Marks 05

Answer

$$\frac{d^2 y}{dx^2} + 4y = \sin^2 2x$$

$$\frac{d^2 y}{dx^2} + 4y = \frac{1}{2} - \frac{\cos 4x}{2} \quad \left[\begin{array}{l} \because \cos 4x = 1 - 2\sin^2 2x \\ \therefore \sin^2 2x = \frac{1 - \cos 4x}{2} \end{array} \right]$$

The auxiliary equation is,

$$m^2 + 4 = 0$$

$$m = \pm 2i$$

$$CF = C_1 \cos 2x + C_2 \sin 2x$$

$$PI = \frac{1}{D^2 + 4} \sin^2 2x = \frac{1}{2} \left[\frac{1}{D^2 + 4} (1 - \cos 4x) \right]$$

$$= \frac{1}{2} \left[\frac{1}{D^2 + 4} (e^{0x}) - \frac{1}{D^2 + 4} \cos 4x \right]$$

$$= \frac{1}{2} \left[\frac{1}{0 + 4} - \frac{1}{-16 + 4} \cos 4x \right]$$

$$= \frac{1}{2} \left[\frac{1}{4} + \frac{1}{12} \cos 4x \right]$$

$$= \frac{1}{8} \left[1 + \frac{1}{3} \cos 4x \right]$$

Now the complete solution is,

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{8} \left[1 + \frac{1}{3} \cos 4x \right] \quad \dots(1.4.1)$$

$$y' = -2C_1 \sin 2x + 2C_2 \cos 2x - \frac{1}{6} \sin 4x \quad \dots(1.4.2)$$

Using boundary conditions,

$$y(0) = 0 \text{ and } y'(0) = 0$$

From eq. (1.4.1), we have

$$0 = C_1 + \frac{1}{8} \Rightarrow C_1 = -\frac{1}{8}$$

From eq. (1.4.2), we have

$$0 = 2C_2 \Rightarrow C_2 = 0$$

Putting the value of C_1 and C_2 in eq. (1.4.1), we get

$$y = -\frac{1}{8} \cos 2x + \frac{1}{8} \left[1 + \frac{1}{3} \cos 4x \right]$$

Que 1.5. Solve $\frac{d^2 x}{dt^2} + 9x = \cos 3t$

AKTU 2013-14, Marks 05

Answer

$$\frac{d^2 x}{dt^2} + 9x = \cos 3t$$

$$(D^2 + 9)x = \cos 3t$$

Auxiliary equation, $m^2 + 9 = 0$

$$m^2 = -9 \Rightarrow m = \pm 3i$$

$$CF = (C_1 \cos 3t + C_2 \sin 3t)$$

$$PI = \frac{1}{D^2 + 9} \cos 3t$$

$$PI = t \frac{1}{2D} \cos 3t = \frac{t}{2} \left(\frac{\sin 3t}{3} \right) = \frac{t \sin 3t}{6}$$

$$\text{Complete solution, } x = CF + PI = C_1 \cos 3t + C_2 \sin 3t + \frac{t}{6} \sin 3t$$

Que 1.6. Find the particular solution of the differential equation

$$\frac{d^2 y}{dx^2} + a^2 y = \sec ax$$

AKTU 2016-17, Marks 07

Answer

Auxiliary equation is,

$$m^2 + a^2 = 0$$

$$m = \pm ai$$

$$CF = C_1 \cos ax + C_2 \sin ax$$

$$PI = \frac{1}{D^2 + a^2} \sec ax$$

$$= \frac{1}{(D - ia)(D + ia)} \sec ax$$

$$= \frac{1}{2ia} \left[\frac{1}{D - ia} - \frac{1}{D + ia} \right] \sec ax$$

$$= \frac{1}{2ia} \left[\frac{1}{(D - ia)} \sec ax - \frac{1}{(D + ia)} \sec ax \right]$$

$$= \frac{1}{2ia} [P_1 - P_2]$$

Where,

$$P_1 = \frac{1}{D - ia} \sec ax$$

$$= e^{iax} \int e^{-iax} \sec ax \, dx$$

$$= e^{iax} \int (\cos ax - i \sin ax) \sec ax \, dx$$

$$= e^{iax} \int (1 - i \tan ax) \, dx$$

$$= e^{iax} \left\{ x + i \left(\frac{\log \cos ax}{a} \right) \right\}$$

Similarly,

$$P_2 = \frac{1}{D + ia} (\sec ax) = e^{-iax} \left\{ x - i \left(\frac{\log \cos ax}{a} \right) \right\}$$

(Replacing i by $-i$)

$$\begin{aligned} \therefore PI &= \frac{1}{2ia} \left[e^{iax} \left\{ x + i \left(\frac{\log \cos ax}{a} \right) \right\} - e^{-iax} \left\{ x - i \left(\frac{\log \cos ax}{a} \right) \right\} \right] \\ &= \frac{1}{2ia} \left[x(e^{iax} - e^{-iax}) + i \left(\frac{\log \cos ax}{a} \right) (e^{iax} + e^{-iax}) \right] \\ &= \frac{1}{2ia} \left[2ix \sin ax + \frac{i}{a} (\log \cos ax) (2 \cos ax) \right] \\ &= \frac{1}{a} \left[x \sin ax + \frac{1}{a} \cos ax \log \cos ax \right] \end{aligned}$$

Que 1.7. Solve $(D^2 - 2D + 1)y = e^x \sin x$

AKTU 2016-17, Marks 7.5

Answer

$$(D^2 - 2D + 1)y = e^x \sin x$$

Auxiliary equation,

$$m^2 - 2m + 1 = 0$$

$$(m - 1)^2 = 0$$

$$m = 1, 1$$

$$CF = (C_1 + C_2 x)e^x$$

$$PI = \frac{1}{(D^2 - 2D + 1)} e^x \sin x$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} \sin x$$

$$= e^x \frac{1}{(D^2 + 2D + 1 - 2D - 2 + 1)} \sin x = e^x \frac{\sin x}{D^2}$$

Replace D^2 by -1

$$= -e^x \sin x$$

$\therefore$ Complete solution = CF + PI

$$y = (C_1 + C_2 x)e^x - e^x \sin x$$

Que 1.8. Solve: $(D^2 - 3D + 2)y = x^2 + 2x + 1$.

AKTU 2014-15, Marks 05

Answer

$$(D^2 - 3D + 2)y = x^2 + 2x + 1$$

Auxiliary equation,

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

$$CF = C_1 e^x + C_2 e^{2x}$$

$$PI = \frac{1}{(D^2 - 3D + 2)} (x^2 + 2x + 1)$$

$$= \frac{1}{2} \left[1 + \frac{D^2 - 3D}{2} \right]^{-1} (x^2 + 2x + 1)$$

$$= \frac{1}{2} \left[1 - \frac{D^2}{2} + \frac{3D}{2} + \left(\frac{D^2 - 3D}{2} \right)^2 - \dots \right] (x^2 + 2x + 1)$$

$$= \frac{1}{2} \left[1 - \frac{D^2}{2} + \frac{3D}{2} + \frac{9D^2}{4} \right] (x^2 + 2x + 1)$$

(Neglecting higher terms)

$$= \frac{1}{2} \left[x^2 + 2x + 1 - \frac{2}{2} + \frac{3}{2} (2x + 2) + \frac{9}{4} \times 2 \right]$$

$$= \frac{1}{2} \left[x^2 + 2x + 3x + 3 + \frac{9}{2} \right]$$

$$= \frac{1}{2} \left[x^2 + 5x + \frac{15}{2} \right]$$

$$y = CF + PI$$

$$= C_1 e^x + C_2 e^{2x} + \frac{1}{2} \left[x^2 + 5x + \frac{15}{2} \right]$$

Que 1.9. Solve the differential equation $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = x e^x \cos x$.

AKTU 2013-14, Marks 05

Answer

$$\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = x e^x \cos x$$

Auxiliary equation,

$$(m^2 - 2m + 1) = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$CF = (C_1 + C_2 x) e^x$$

$$PI = \frac{1}{(D-1)^2} x e^x \cos x = \frac{1}{(D-1)^2} e^x (x \cos x)$$

$$= e^x \frac{1}{(D+1-1)^2} x \cos x$$

1-10 F (Sem-2)

Ordinary Differential Equation of Higher Order

$$= e^x \frac{1}{D^2} x \cos x = e^x \frac{1}{D} [x \sin x + \cos x]$$

$$= e^x [-x \cos x + \sin x + \sin x]$$

$$PI = e^x [-x \cos x + 2 \sin x]$$

Complete solution is given by,

$$y = CF + PI$$

$$y = (C_1 + C_2 x) e^x + e^x (-x \cos x + 2 \sin x)$$

Que 1.10. Solve $(D^2 - 2D + 4)y = e^x \cos x + \sin x \cos 3x$.

AKTU 2017-18, Marks 07

Answer

Given equation, $(D^2 - 2D + 4)y = e^x \cos x + \sin x \cos 3x$

Auxiliary equation,

$$m^2 - 2m + 4 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$m = \frac{2 \pm \sqrt{-12}}{2}$$

$$m = 1 \pm i\sqrt{3}$$

Complementary function is,

$$CF = e^x (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x)$$

Particular integral, $PI = P_1 + P_2$

$$P_1 = e^x \cos x$$

$$= \frac{1}{D^2 - 2D + 4} e^x \cos x$$

[Replace D by $D+1$]

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^x \frac{1}{D^2 + 3} \cos x$$

[Replace D^2 by -1]

$$= e^x \frac{1}{-1 + 3} \cos x$$

$$= e^x \frac{\cos x}{2}$$

$$P_2 = \frac{1}{D^2 - 2D + 4} \sin x \cos 3x$$

$$\begin{aligned}
&= \frac{1}{2} \frac{1}{(D^2 - 2D + 4)} 2 \sin x \cos 3x \\
&= \frac{1}{2} \frac{1}{(D^2 - 2D + 4)} (\sin(x + 3x) + \sin(x - 3x)) \\
&= \frac{1}{2} \frac{1}{(D^2 - 2D + 4)} (\sin 4x - \sin 2x) \\
&= \frac{1}{2} \left[\frac{1}{D^2 - 2D + 4} \sin 4x - \frac{1}{D^2 - 2D + 4} \sin 2x \right] \\
&= \frac{1}{2} \left[\frac{1}{-(4)^2 - 2D + 4} \sin 4x - \frac{1}{-(2)^2 - 2D + 4} \sin 2x \right] \\
&= \frac{1}{2} \left[\frac{1}{-12 - 2D} \sin 4x - \frac{1}{-2D} \sin 2x \right] \\
&= \frac{1}{4} \left[\frac{-1}{D + 6} \sin 4x + \frac{1}{D} \sin 2x \right] \\
&= \frac{1}{4} \left[\frac{-(D - 6)}{D^2 - 36} \sin 4x - \frac{\cos 2x}{2} \right] \\
&= \frac{1}{4} \left[\frac{-(D - 6)}{-52} \sin 4x - \frac{\cos 2x}{2} \right] \\
&= \frac{1}{4} \left[\frac{4 \cos 4x - 6 \sin 4x}{52} - \frac{\cos 2x}{2} \right] \\
&= \frac{1}{4} \left[\frac{4 \cos 4x - 6 \sin 4x}{52} \right] - \frac{\cos 2x}{8}
\end{aligned}$$

Complete solution,

$$\begin{aligned}
y &= CF + PI \\
&= CF + P_1 + P_2 \\
y &= e^x (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x) + e^x \frac{\cos x}{2} \\
&\quad + \frac{1}{4} \left[\frac{4 \cos 4x - 6 \sin 4x}{52} \right] - \frac{\cos 2x}{8}
\end{aligned}$$

PART-2**Simultaneous Linear Differential Equations.****CONCEPT OUTLINE**

Simultaneous Differential Equation : If two or more dependent variables are functions of a single independent variable, the equations which consist of the derivatives of such variables are called simultaneous differential equations.

Questions-Answers**Long Answer Type and Medium Answer Type Questions**

Que 1.11. Solve the simultaneous equation $\frac{dx}{dt} + 5x - 2y = t$,

$\frac{dy}{dt} + x + y = 0$ being given $x = 0, y = 0$ when $t = 0$.

AKTU 2014-15, Marks 10**Answer**

$$(D + 5)x - 2y = t \quad \dots(1.11.1)$$

$$x + (D + 1)y = 0 \quad \dots(1.11.2)$$

On multiplying eq. (1.11.2) by $(D + 5)$ and subtracting from eq. (1.11.1), we get

$$(D + 1)(D + 5)y + 2y = -t$$

$$(D^2 + 6D + 5 + 2)y = -t$$

$$\text{Auxiliary equation, } m^2 + 6m + 7 = 0$$

$$m = \frac{-6 \pm \sqrt{36 - 28}}{2} \Rightarrow m = -3 \pm \sqrt{2}$$

$$CF = e^{-3t} (C_1 \cosh \sqrt{2}t + C_2 \sinh \sqrt{2}t)$$

$$PI = \frac{1}{D^2 + 6D + 7} (-t)$$

$$= \frac{-1}{7} \left(1 + \frac{D^2 + 6D}{7} \right)^{-1} (t) = -\frac{1}{7} \left(1 - \frac{6D}{7} \right) t$$

$$PI = -\frac{1}{7} \left(t - \frac{6}{7} \right)$$

$$y = e^{-3t} (C_1 \cosh \sqrt{2}t + C_2 \sinh \sqrt{2}t) - \frac{1}{7} \left(t - \frac{6}{7} \right) \dots(1.11.3)$$

$$\frac{dy}{dt} = e^{-3t} (-C_1 \sqrt{2} \sinh \sqrt{2}t + \sqrt{2} C_2 \cosh \sqrt{2}t)$$

$$-3e^{-3t} (C_1 \cosh \sqrt{2}t + C_2 \sinh \sqrt{2}t) - \frac{1}{7}$$

From eq. (1.11.2),

$$x = -\frac{dy}{dt} - y$$

$$\begin{aligned}
 x &= -e^{-3t} (-C_1 \sqrt{2} \sinh \sqrt{2} t + \sqrt{2} C_2 \cosh \sqrt{2} t) + \frac{1}{7} \\
 &\quad + 3e^{-3t} (C_1 \cosh \sqrt{2} t + C_2 \sinh \sqrt{2} t) \\
 &\quad - e^{-3t} (C_1 \cosh \sqrt{2} t + C_2 \sinh \sqrt{2} t) + \frac{1}{7} \left(t - \frac{6}{7} \right) \\
 x &= -e^{-3t} (-C_1 \sqrt{2} \sinh \sqrt{2} t + \sqrt{2} C_2 \cosh \sqrt{2} t) \\
 &\quad + 2e^{-3t} (C_1 \cosh \sqrt{2} t + C_2 \sinh \sqrt{2} t) + \frac{t}{7} + \frac{1}{49} \dots (1.11.4)
 \end{aligned}$$

Boundary conditions

$$x(0) = 0, y(0) = 0$$

From eq. (1.11.3) and eq. (1.11.4), we have

$$0 = C_1 + \frac{6}{7}$$

$$C_1 = -\frac{6}{7}$$

$$\text{and } 0 = -\sqrt{2} C_2 + 2C_1 + \frac{1}{49}$$

$$\sqrt{2} C_2 = -\frac{12}{7} + \frac{1}{49}$$

$$\sqrt{2} C_2 = -\frac{83}{49}$$

$$C_2 = -\frac{83}{49\sqrt{2}}$$

Now,

$$y = e^{-3t} \left[-\frac{6}{7} \cosh \sqrt{2} t - \frac{83}{49\sqrt{2}} \sinh \sqrt{2} t \right] - \frac{1}{7} \left(t - \frac{6}{7} \right)$$

$$\begin{aligned}
 x &= -e^{-3t} \left(-\frac{6}{7} \sqrt{2} \sinh \sqrt{2} t - \frac{83}{49} \cosh \sqrt{2} t \right) \\
 &\quad + 2e^{-3t} \left(t - \frac{6}{7} \sqrt{2} t - \frac{83}{49\sqrt{2}} \sinh \sqrt{2} t \right) + \frac{t}{7} + \frac{1}{49}
 \end{aligned}$$

Que 1.12. Solve the following simultaneous equations.

$$\frac{d^2 x}{dt^2} + y = \sin t$$

$$\frac{d^2 y}{dt^2} + x = \cos t$$

AKTU 2015-16, Marks 10

Answer

Let $\frac{d}{dt} = D$ then the given system of equations become

$$D^2 x + y = \sin t \quad \dots (1.12.1)$$

$$x + D^2 y = \cos t \quad \dots (1.12.2)$$

Multiplying eq. (1.12.1) by D^2 , we get

$$D^4 x + D^2 y = -\sin t \quad \dots (1.12.3)$$

Subtracting eq. (1.12.2) from eq. (1.12.3), we get

$$(D^4 - 1)x = -\sin t - \cos t$$

Auxiliary equation is

$$m^4 - 1 = 0$$

$$(m^2 - 1)(m^2 + 1) = 0$$

$$\Rightarrow m = 1, -1, \pm i$$

$$\therefore \text{CF} = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t$$

$$\text{PI} = \frac{1}{D^4 - 1} (-\sin t - \cos t)$$

$$= -t \frac{1}{4D^3} (\sin t + \cos t) = \frac{t}{4} (-\cos t + \sin t)$$

$$\therefore x = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t + \frac{t}{4} (\sin t - \cos t) \quad \dots (1.12.4)$$

$$Dx = C_1 e^t - C_2 e^{-t} - C_3 \sin t + C_4 \cos t + \frac{t}{4} (\cos t + \sin t) + \frac{1}{4} (\sin t - \cos t)$$

$$\begin{aligned}
 D^2 x &= C_1 e^t + C_2 e^{-t} - C_3 \cos t - C_4 \sin t + \frac{t}{4} (-\sin t + \cos t) \\
 &\quad + \frac{1}{4} (\cos t + \sin t) + \frac{1}{4} (\cos t + \sin t)
 \end{aligned}$$

$$\text{From eq. (1.12.1), } y = \sin t - \frac{d^2 x}{dt^2}$$

$$y = \sin t - C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t + \frac{t}{4} (\sin t - \cos t) - \frac{1}{2} (\sin t + \cos t) \quad \dots (1.12.5)$$

Eq. (1.12.4) and eq. (1.12.5), when taken together give the complete solution of the given system of equations.

Que 1.13. Solve the following :

$$\frac{dx}{dt} = 3x + 8y$$

$$\frac{dy}{dt} = -x - 3y \text{ with } x(0) = 6 \text{ and } y(0) = -2$$

AKTU 2013-14, Marks 05

Answer

$$\frac{dx}{dt} = 3x + 8y$$

$$\frac{dy}{dt} = -x - 3y$$

Let $\frac{d}{dt} = D$, so the given equation reduces to

$$(D - 3)x - 8y = 0 \quad \dots(1.13.1)$$

$$x + (D + 3)y = 0 \quad \dots(1.13.2)$$

Multiply by $(D + 3)$ in eq. (1.13.1) and multiply by 8 in eq. (1.13.2), then adding both equations, we get

$$(D^2 - 9 + 8)x = 0$$

$$(D^2 - 1)x = 0$$

Auxiliary equation is, $m^2 - 1 = 0$

$$m = \pm 1$$

$$CF = C_1 e^{-t} + C_2 e^t$$

$$PI = 0$$

$$x = C_1 e^{-t} + C_2 e^t \quad \dots(1.13.3)$$

From eq. (1.13.1),

$$8y(t) = \frac{dx(t)}{dt} - 3x(t)$$

or

$$8y = \frac{dx}{dt} - 3x$$

$$8y = C_1(-1)e^{-t} + C_2 e^t - 3[C_1 e^{-t} + C_2 e^t]$$

$$8y = -4C_1 e^{-t} - 2C_2 e^t$$

$$y = -0.5 C_1 e^{-t} - 0.25 C_2 e^t \quad \dots(1.13.4)$$

Apply boundary condition,

$$x(0) = 6$$

$$\text{From eq. (1.13.3), } 6 = C_1 + C_2 \quad \dots(1.13.5)$$

$$\text{From eq. (1.13.4), } y(0) = -2 = -0.5 C_1 - 0.25 C_2 \quad \dots(1.13.6)$$

By solving eq. (1.13.5) and eq. (1.13.6), we get

$$C_1 = 2$$

$$C_2 = 4$$

$$x = 2e^{-t} + 4e^t$$

$$y = -e^{-t} - e^t$$

Que 1.14. Solve: $\frac{d^2 x}{dt^2} + \frac{dy}{dt} + 3x = e^{-t}$, $\frac{d^2 y}{dt^2} - 4\frac{dx}{dt} + 3y = \sin 2t$.

AKTU 2021-22, Marks 10

Answer

Let $D = \frac{d}{dt}$ then we have

$$(D^2 + 3)x + Dy = e^{-t} \quad \dots(1.14.1)$$

$$-4Dx + (D^2 + 3)y = \sin 2t \quad \dots(1.14.2)$$

Operating eq. (1.14.1) by $(D^2 + 3)$ and eq. (1.14.2) by D then subtracting, we have

$$[(D^2 + 3)^2 + 4D^2]x = 4e^{-t} - 2\cos 2t$$

$$(D^4 + 10D^2 + 9)x = 4e^{-t} - 2\cos 2t$$

Auxiliary equation is,

$$m^4 + 10m^2 + 9 = 0 \Rightarrow m = \pm i, \pm 3i$$

$$CF = c_1 \cos t + c_2 \sin t + c_3 \cos 3t + c_4 \sin 3t$$

$$PI = \frac{1}{D^4 + 10D^2 + 9} (4e^{-t}) - \frac{1}{D^4 + 10D^2 + 9} (2\cos 2t)$$

$$= \frac{1}{1 + 10 + 9} (4e^{-t}) - \frac{1}{16 - 40 + 9} (2\cos 2t)$$

$$= \frac{1}{5} e^{-t} + \frac{2}{15} \cos 2t$$

$$x = c_1 \cos t + c_2 \sin t + c_3 \cos 3t + c_4 \sin 3t$$

$$+ \frac{1}{5} e^{-t} + \frac{2}{15} \cos 2t \quad \dots(1.14.3)$$

Again operating eq. (1.14.1) by $4D$ and eq. (2) by $(D^2 + 3)$ then adding, we get

$$[(D^2 + 3)^2 + 4D^2]y = -4e^{-t} - \sin 2t$$

$$(D^4 + 10D^2 + 9)y = -4e^{-t} - \sin 2t$$

Auxiliary equation is,

$$m^4 + 10m^2 + 9 = 0 \Rightarrow m = \pm i, \pm 3i$$

$$CF = c_5 \cos t + c_6 \sin t + c_7 \cos 3t + c_8 \sin 3t$$

$$PI = \frac{1}{D^4 + 10D^2 + 9} (-4e^{-t}) - \frac{1}{D^4 + 10D^2 + 9} (\sin 2t)$$

$$= -\frac{1}{5}e^{-t} + \frac{1}{15}\sin 2t$$

$$x = c_5 \cos t + c_6 \sin t + c_7 \cos 3t + c_8 \sin 3t$$

$$-\frac{1}{5}e^{-t} + \frac{1}{15}\sin 2t \quad \dots(1.14.4)$$

Equation (1.14.3) and (1.14.4), when taken together, give the complete solution.

Que 1.15. Solve the simultaneous differential equations

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = y \text{ and } \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 25x + 16e^t$$

AKTU 2017-18, Marks 07

Answer

The given equations can be written as given below :

$$(D^2 - 4D + 4)x - y = 0 \quad \dots(1.15.1)$$

$$-25x + (D^2 + 4D + 4)y = 16e^t \quad \dots(1.15.2)$$

Multiplying eq. (1.15.1) by $D^2 + 4D + 4$ and adding to eq. (1.15.2), we get

$$(D^2 - 4D + 4)(D^2 + 4D + 4)x - 25x = 16e^t$$

$$(D^4 - 8D^2 - 9)x = 16e^t$$

Auxiliary equation is,

$$m^4 - 8m^2 - 9 = 0$$

$$(m^2 - 9)(m^2 + 1) = 0 \Rightarrow m = \pm i, \pm 3$$

$$CF = C_1 e^{3t} + C_2 e^{-3t} + C_3 \cos t + C_4 \sin t$$

$$PI = \frac{1}{D^4 - 8D^2 - 9}(16e^t) = -e^t$$

$$x = C_1 e^{3t} + C_2 e^{-3t} + C_3 \cos t + C_4 \sin t - e^t$$

$$\frac{dx}{dt} = 3C_1 e^{3t} - 3C_2 e^{-3t} + C_3(-\sin t) + C_4 \cos t - e^t \quad \dots(1.15.3)$$

$$\frac{d^2x}{dt^2} = 9C_1 e^{3t} + 9C_2 e^{-3t} - C_3 \cos t - C_4 \sin t - e^t$$

From eq. (1.15.1), $y = \frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x$

$$= 9C_1 e^{3t} + 9C_2 e^{-3t} - C_3 \cos t - C_4 \sin t - e^t - 4(3C_1 e^{3t} - 3C_2 e^{-3t} - C_3 \sin t + C_4 \cos t - e^t) + 4(C_1 e^{3t} + C_2 e^{-3t} + C_3 \cos t + C_4 \sin t - e^t)$$

1-18 F (Sem-2)

Ordinary Differential Equation of Higher Order

$$\Rightarrow y = C_1 e^{3t} + 25C_2 e^{-3t} + (3C_3 - 4C_4) \cos t + (4C_3 + 3C_4) \sin t - e^t \quad \dots(1.15.4)$$

Eq. (1.15.3) and eq. (1.15.4) when taken together give the complete solution.

Que 1.16. Solve $\frac{dx}{dt} + 2x + 4y = 1 + 4t$; $\frac{dy}{dt} + x - y = \frac{3}{2}t^2$.

AKTU 2012-13, Marks 05

Answer

$$\frac{dx}{dt} + 2x + 4y = 1 + 4t, \quad \frac{dy}{dt} + x - y = \frac{3}{2}t^2$$

Writing D for $\frac{d}{dt}$, the given equation becomes

$$-(D + 2)x + 4y = 1 + 4t \quad \dots(1.16.1)$$

$$x + (D - 1)y = \frac{3}{2}t^2 \quad \dots(1.16.2)$$

To eliminate y , multiplying eq. (1.16.1) by $(D - 1)$ and multiplying eq. (1.16.2) by 4, then subtracting, we get

$$[(D + 2)(D - 1) - 4]x = (D - 1)(1 + 4t) - 4\left(\frac{3}{2}t^2\right)$$

$$(D^2 + 2D - D - 2 - 4)x = -1 + 4t - 6t^2$$

$$(D^2 + D - 6)x = -1 + 4t - 6t^2$$

Auxiliary equation is,

$$m^2 + m - 6 = 0$$

$$m^2 + 3m - 2m - 6 = 0$$

$$m(m + 3) - 2(m + 3) = 0$$

$$(m + 3)(m - 2) = 0 \Rightarrow m = 2, -3$$

$$\therefore CF = C_1 e^{2t} + C_2 e^{-3t}$$

$$PI = \frac{1}{(D^2 + D - 6)}(-1 + 4t - 6t^2)$$

$$= \frac{3}{(D^2 + D - 6)}e^{0t} - \frac{4t}{(D^2 + D - 6)} - \frac{6t^2}{(D^2 + D - 6)}$$

$$= -\frac{3}{6} + \frac{4}{6} \left[1 + \left(-\frac{D^2}{6} - \frac{D}{6} \right) \right]^{-1} t + \frac{6}{6} \left[1 + \left(-\frac{D^2}{6} - \frac{D}{6} \right) \right]^{-1} t^2$$

$$= -\frac{3}{6} + \frac{4}{6} \left[1 + \left(-\frac{D}{6} - \frac{D^2}{6} \right) \right]^{-1} t + \left[1 + \left(-\frac{D}{6} - \frac{D^2}{6} \right) \right]^{-1} t^2$$

$$= -\frac{3}{6} + \frac{4}{6} \left[1 + \frac{D}{6} + \frac{D^2}{6} \right] t + \left[1 - \left(-\frac{D}{6} - \frac{D^2}{6} \right) + \left(-\frac{D}{6} - \frac{D^2}{6} \right)^2 \right] t^2$$

$$= -\frac{3}{6} + \frac{4t}{6} + \frac{4}{36} + t^2 + \frac{2t}{6} + \frac{2}{6} + \frac{2}{36} = t^2 + \frac{6t}{6} + \frac{(-18 + 4 + 12 + 2)}{36}$$

So, $PI = t^2 + t$
 $x = C_1 e^{2t} + C_2 e^{-3t} + t^2 + t$

Now $\frac{dx}{dt} = 2C_1 e^{2t} - 3C_2 e^{-3t} + 2t + 1$

Substituting the values of x and $\frac{dx}{dt}$ in eq. (1.16.1), we get

$$4y = -2C_1 e^{2t} + 3C_2 e^{-3t} - 2t - 1 - 2C_1 e^{2t} - 2C_2 e^{-3t} - 2t^2 - 2t + 1 + 4t$$

$$y = -C_1 e^{2t} + \frac{1}{4} C_2 e^{-3t} - \frac{1}{2} t^2$$

PART-3

Second Order Linear Differential Equations with Variable Coefficients, Solution by Changing Independent Variable.

CONCEPT OUTLINE

Second Order Linear Differential Equation : A differential equation of the form $\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R$ is known linear differential equation of second order, where P , Q and R are functions of x alone. The homogeneous linear equations :

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 x \frac{dy}{dx} + a_0 y = 0 \quad \dots(1)$$

where $a_0, a_1, a_2, \dots$ are constants, is called a homogeneous equation.

Put $x = e^z$, $z = \log_e x$, $\frac{d}{dx} = D$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$$

or $x \frac{dy}{dx} = \frac{dy}{dz}$ or $x \frac{dy}{dx} = Dy$

Again, $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dz} \right)$
 $= -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d^2 y}{dz^2} \frac{dz}{dx}$

$$= -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d^2 y}{dz^2} \cdot \frac{1}{x}$$

$$= \frac{1}{x^2} \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right) = \frac{1}{x^2} (D^2 - D) y$$

$$x^2 \frac{d^2 y}{dx^2} = (D^2 - D)y \quad \text{or} \quad x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

Similarly, $x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$

The substitution of these values in eq. (1) reduces the given homogeneous equation to a differential equation with constant coefficients.

Legendre's linear differential equation :

An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} (a+bx) \frac{dy}{dx} + a_n y = Q$$

where a_i 's are constants and Q is a function of x , is called Legendre's linear differential equation. Such equations can be reduced to linear differential equations with constant co-efficient by the substitution

$$a+bx = e^z \text{ i.e., } z = \log(a+bx) \text{ so that } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{b}{a+bx} \frac{dy}{dz}$$

$$\text{or } (a+bx) \frac{dy}{dx} = b \frac{dy}{dz} = b Dy, \text{ where } D = \frac{d}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{b}{a+bx} \frac{dy}{dz} \right) = -\frac{b^2}{(a+bx)^2} \frac{dy}{dz} + \frac{b}{a+bx} \frac{d^2 y}{dz^2} \frac{dz}{dx}$$

$$= -\frac{b^2}{(a+bx)^2} \frac{dy}{dz} + \frac{b}{a+bx} \frac{d^2 y}{dz^2} \cdot \frac{b}{a+bx} = \frac{b^2}{(a+bx)^2} \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right)$$

$$\text{or } (a+bx)^2 \frac{d^2 y}{dx^2} = b^2 (D^2 y - Dy) = b^2 D(D-1)y$$

$$\text{Similarly, } (a+bx)^3 \frac{d^3 y}{dx^3} = b^3 D(D-1)(D-2)y.$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.17. Solve $(3x+2)^2 \frac{d^2y}{dx^2} - (3x+2) \frac{dy}{dx} - 12y = 6x$.

Answer

$$(3x+2)^2 \frac{d^2y}{dx^2} - (3x+2) \frac{dy}{dx} - 12y = 6x$$

Put $3x+2 = e^z$ so that $z = \log(3x+2)$

$$\text{Also, } (3x+2) \frac{dy}{dx} = 3Dy,$$

$$(3x+2)^2 \frac{d^2y}{dx^2} = 9D(D-1)y$$

$$\text{and } D \equiv \frac{d}{dz}$$

The given equation becomes,

$$9D(D-1)y - 3Dy - 12y = 2(e^z - 2)$$

$$(9D^2 - 9D - 3D - 12)y = 2(e^z - 2)$$

The auxiliary equation is

$$9m^2 - 12m - 12 = 0$$

$$(m-2)\left(m+\frac{2}{3}\right) = 0$$

$$m = 2, -\frac{2}{3}$$

Therefore, the complementary function is

$$CF = C_1 e^{2z} + C_2 e^{-\frac{2z}{3}}$$

and

$$PI = \frac{1}{9D^2 - 12D + 12} 2(e^z - 2)$$

$$= 2 \left\{ \frac{1}{9D^2 - 12D - 12} e^z - 2 \frac{e^0}{9D^2 - 12D - 12} \right\}$$

$$= 2 \left\{ \frac{1}{9-12-12} e^z - 4 \frac{1}{0-0-12} \right\} = \frac{2e^z}{-15} + \frac{1}{3}$$

The solution is,

$$y = CF + PI$$

$$y = C_1 e^{2z} + C_2 e^{-\frac{2z}{3}} + \frac{1}{3} - \frac{2}{15} e^z$$

$$= C_1 (3x+2)^2 + C_2 (3x+2)^{-2/3} + \frac{1}{3} - \frac{2}{15} (3x+2)$$

$$(\because e^z = 3x+2)$$

C_1 and C_2 are arbitrary constants of integration.

Que 1.18. Solve: $(1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4 \cos \log(1+x)$.

AKTU 2021-22, Marks 10

Answer

$$\text{Put } 1+x = e^z$$

So that $z = \log(1+x)$ and let $D \equiv \frac{d}{dz}$

$$\text{Also, } (1+x) \frac{dy}{dx} = Dy, (1+x)^2 \frac{d^2y}{dx^2} = D(D-1)y$$

The given equation becomes,

$$[D(D-1) + D + 1]y = 4 \cos z$$

$$(D^2 + 1)y = 4 \cos z$$

Auxiliary equation is,

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\therefore CF = c_1 \cos z + c_2 \sin z$$

$$PI = \frac{1}{D^2 + 1} (4 \cos z) = 4x \cdot \frac{1}{2D} \cos z = 2z \sin z$$

Hence the complete solution is,

$$y = CF + PI$$

$$y = c_1 \cos z + c_2 \sin z + 2z \sin z$$

$$= c_1 \cos(\log(1+x)) + c_2 \sin(\log(1+x)) + 2 \log(1+x) \sin(\log(1+x)) \quad [\because z = \log(1+x)]$$

where, c_1 and c_2 are arbitrary constants of integration.

Que 1.19. Solve $\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} = \frac{12 \log x}{x}$

Answer

Given equation may be written as

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 12 \log x$$

$$\text{or } (D(D-1) + D)y = 12z$$

$$(\text{Let, } z = \log x)$$

$$D^2y = 12z$$

Auxiliary equation is, $m^2 = 0$

$$m = 0, 0$$

$$CF = (C_1 + C_2 z) e^{0z} = C_1 + C_2 z$$

$$PI = \frac{1}{D^2} 12z = 12 \frac{1}{D^2} z = 12 \frac{z^3}{6} = 2z^3$$

Complete solution, $y = CF + PI$

$$y = C_1 + C_2 z + 2z^3$$

$$y = C_1 + C_2 \log x + 2(\log x)^3$$

Que 1.20. Write the procedure for solving the linear differential equation by changing the independent variable.

Answer

Following are procedure for solving the linear differential equation by changing the independent variable :

1. Make the coefficient of $\frac{d^2 y}{dx^2}$ as 1 if it is not so.
2. Compare the equation with standard form $y'' + Py' + Qy = R$ and get P , Q and R .

3. Choose z such that $\left(\frac{dz}{dx}\right)^2 = Q$

Here Q is taken in such a way that it remains the whole square of a function without surd and its negative sign is ignored.

4. Find $\frac{dz}{dx}$ hence obtain z (on integration) and $\frac{d^2 z}{dx^2}$ (on differentiation).
5. Find P_1 , Q_1 and R_1 by the formulae,

$$P_1 = \frac{\frac{d^2 z}{dx^2} + P \frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

6. Reduced equation is $\frac{d^2 y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$ which we solve for y in terms of z .
7. We write the complete solution as y in terms of x by replacing the value of z in terms of x .

Que 1.21. Solve by changing the independent variable :

$$\frac{d^2 y}{dx^2} + (3 \sin x - \cot x) \frac{dy}{dx} + 2y \sin^2 x = e^{-\cos x} \sin^2 x$$

AKTU 2014-15, Marks 05

Answer

On comparing given equation with $y'' + Py' + Qy = R$, we have

$$P = 3 \sin x - \cot x, Q = 2 \sin^2 x \text{ and } R = e^{-\cos x} \sin^2 x$$

On changing the independent variable x to z , we get

$$\frac{d^2 y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

$$\text{where, } P_1 = \frac{\frac{d^2 z}{dx^2} + P \frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Let us take $Q_1 = \text{constant}$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{2 \sin^2 x}{\left(\frac{dz}{dx}\right)^2} = \text{constant} = 2 \text{ (say)}$$

$$\left(\frac{dz}{dx}\right)^2 = \sin^2 x \Rightarrow \frac{dz}{dx} = \sin x \quad \dots(1.21.1)$$

Integration yields, $z = -\cos x \quad \dots(1.21.2)$

$$\text{From eq. (1.21.1), } \frac{d^2 z}{dx^2} = \cos x$$

$$\therefore P_1 = \frac{\cos x + (3 \sin x - \cot x) \sin x}{\sin^2 x} = 3$$

$$Q_1 = \frac{2 \sin^2 x}{\sin^2 x} = 2$$

$$R_1 = \frac{e^{-\cos x} \sin^2 x}{\sin^2 x} = e^{-\cos x} = e^z$$

Reduced equation is, $\frac{d^2 y}{dz^2} + 3 \frac{dy}{dz} + 2y = e^z$

Auxiliary equation, is $m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$

$$\therefore CF = C_1 e^{-z} + C_2 e^{-2z}$$

$$PI = \frac{1}{D^2 + 3D + 2} (e^z) = \frac{e^z}{6}$$

Complete solution is, $y = C_1 e^{-z} + C_2 e^{-2z} + \frac{e^z}{6} \quad (\because z = -\cos x)$

$$y = C_1 e^{\cos x} + C_2 e^{2 \cos x} + \frac{e^{-\cos x}}{6}$$

PART-4*Method of Variation of Parameters.***CONCEPT OUTLINE**

Method of Variation of Parameters : By this method the general solution is obtained by varying the arbitrary constants of the complementary function that is why the method is known as method of variation of parameters.

Procedure :

1. Find out the parts of CF.
2. Let them be u and v .
3. Consider $y = Au + Bv$ as the complete solution of equation given.
4. A and B are determined by the formulae

$$A = \int \frac{-Rv}{uv_1 - u_1v} dx + c_1$$

and

$$B = \int \frac{Ru}{uv_1 - u_1v} dx + c_2$$

where C_1 and C_2 are the arbitrary constant of integration.

5. Write $y = Au + Bv$ as the complete solution.

Questions-Answers**Long Answer Type and Medium Answer Type Questions**

Que 1.22. Using variation of parameters method, solve

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - 12y = 0$$

AKTU 2015-16, Marks 10

Answer

Consider the equation $x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - 12y = 0$

Put $x = e^z$ so that $z = \log x$ and Let $D \equiv \frac{d}{dz}$ then the given equation

reduces to,

$$[D(D-1) + 2D - 12]y = 0$$

$$\Rightarrow (D^2 + D - 12)y = 0$$

Auxiliary equation is,

$$m^2 + m - 12 = 0 \Rightarrow m = 3, -4$$

$$\therefore \text{CF} = C_1 e^{3x} + C_2 e^{-4x} = C_1 x^3 + C_2 x^{-4}$$

$$\text{PI} = 0$$

The complete solution is given as,

$$y = C_1 x^3 + C_2 x^{-4}$$

Que 1.23. Solve $\frac{d^2 y}{dx^2} + y = \tan x$ by method of variation of

parameters.

AKTU 2018-19, Marks 10

Answer

$$(D^2 + 1)y = \tan x$$

Auxiliary equation,

$$m^2 + 1 = 0$$

$$m = \sqrt{-1} = \pm i$$

$$\text{CF} = C_1 \cos x + C_2 \sin x$$

Here,

$$u = \cos x \text{ and}$$

$$v = \sin x \text{ are two parts of CF}$$

Let $y = A \cos x + B \sin x$ be the complete solution of the given equation where A and B are determined as:

$$A = - \int \frac{Rv}{uv_1 - u_1v} dx + C_1$$

$$= - \int \frac{\tan x \sin x}{\cos^2 x + \sin^2 x} dx + C_1 \quad (\because R = \tan x)$$

$$= - \int \frac{1 - \cos^2 x}{\cos x} dx + C_1$$

$$= - \int (\sec x - \cos x) dx + C_1$$

$$A = \sin x - \log (\sec x + \tan x) + C_1$$

and

$$B = \int \frac{Ru}{uv_1 - u_1v} dx + C_2 = \int \frac{\tan x \cos x}{1} dx + C_2$$

$$B = -\cos x + C_2$$

Hence the complete solution is,

$$y = A \cos x + B \sin x$$

$$= [\sin x - \log (\sec x + \tan x) + C_1] \cos x + (-\cos x + C_2) \sin x$$

$$y = C_1 \cos x + C_2 \sin x - \cos x \log (\sec x + \tan x)$$

Que 1.24. Use the variation of parameter method to solve the differential equation $(D^2 - 1)y = 2(1 - e^{-2x})^{-1/2}$

AKTU 2021-22, Marks 10

Answer

$$(D^2 - 1)y = 2(1 - e^{-2x})^{-1/2}$$

Auxiliary equation,

$$m^2 - 1 = 0$$

$$m = \sqrt{1} = \pm 1$$

$$CF = C_1 e^x + C_2 e^{-x}$$

Here

 $u = e^x$ and $v = e^{-x}$ are two parts of CFLet $y = A e^x + B e^{-x}$ be the complete solution of the given equation where, A and B are determined as :

$$\begin{aligned} A &= - \int \frac{Rv}{uv_1 - u_1v} dx + c_1 \\ &= - \int \frac{2(1 - e^{-2x})^{-1/2} e^{-x}}{e^x(-e^{-x}) - e^x e^{-x}} dx + c_1 \\ &= - \int \frac{2(1 - e^{-2x})^{-1/2}}{e^x(-e^{-x}) - e^x e^{-x}} dx + c_1 \quad [\because R = 2(1 - e^{-2x})^{-1/2}] \end{aligned}$$

$$\begin{aligned} &= -2 \int \frac{e^{-x}}{-2\sqrt{1 - e^{-2x}}} dx + c_1 \\ &= \int \frac{e^{-x}}{\sqrt{1 - e^{-2x}}} dx + c_1 \quad \text{Put } e^{-x} = t \therefore e^{-x} dx = -dt \end{aligned}$$

$$= - \int \frac{dt}{\sqrt{1 - t^2}} + c_1$$

$$A = -\sin^{-1}(e^{-x}) + c_1$$

$$B = \int \frac{Ru}{uv_1 - u_1v} dx + c_2$$

$$= \int \frac{2(1 - e^{-2x})^{-1/2} e^x}{(-2)} dx + c_2$$

$$= - \int \frac{e^x}{\sqrt{1 - e^{-2x}}} dx + c_2$$

$$= - \int \frac{e^{2x}}{\sqrt{e^{2x} - 1}} dx + c_2 \quad \text{Put } e^{2x} = t \Rightarrow e^{2x} dx = \frac{dt}{2}$$

$$= - \frac{1}{2} \int \frac{dt}{\sqrt{t - 1}} + c_2 = - \frac{1}{2} \frac{(t - 1)^{1/2}}{(1/2)} + c_2$$

$$B = -(e^{2x} - 1)^{1/2} + c_2$$

Hence the complete solution is

$$y = A e^x + B e^{-x}$$

$$y = [-\sin^{-1}(e^{-x}) + c_1] e^x + [-(e^{2x} - 1)^{1/2} + c_2] e^{-x}$$

$$y = c_1 e^x + c_2 e^{-x} - e^x \sin^{-1}(e^{-x}) - e^{-x}(e^{2x} - 1)^{1/2}$$

where, c_1 and c_2 are arbitrary constants of integration.**Que 1.25.** Solve by method of variation of parameters for the differential equation :

$$\frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 9y = \left(\frac{e^{3x}}{x^2}\right)$$

AKTU 2016-17, Marks 07**Answer**

$$(D^2 - 6D + 9)y = \left(\frac{e^{3x}}{x^2}\right)$$

Auxiliary equation,

$$m^2 - 6m + 9 = 0$$

$$(m - 3)^2 = 0$$

$$m = 3, 3$$

So,

$$CF = (C_1 + C_2 x)e^{3x}$$

Here

 $u = e^{3x}$ and $v = x e^{3x}$ are two parts of CF

Also,

$$R = \frac{e^{3x}}{x^2}$$

Let the complete solution be

$$y = A e^{3x} + B x e^{3x}$$

To determine the values of A and B , we have

$$A = \int - \frac{Rv}{uv_1 - u_1v} dx + C_1$$

$$A = \int - \frac{\left(\frac{e^{3x}}{x^2}\right) x e^{3x}}{e^{3x}(e^{3x} + 3x e^{3x}) - x e^{3x} 3e^{3x}} dx + C_1$$

$$A = - \int \frac{e^{6x}/x}{e^{6x}} dx + C_1$$

$$A = - \int \frac{1}{x} dx + C_1$$

$$A = -\log x + C_1$$

$$B = \int \frac{Ru}{uv_1 - u_1v} dx + C_2$$

$$B = \int \frac{\frac{e^{3x}}{x^2} e^{3x}}{e^{3x}(e^{3x} + 3xe^{3x}) - 3e^{3x}xe^{3x}} dx + C_2$$

$$B = \int \frac{1}{x^2} dx + C_2$$

$$B = -\frac{1}{x} + C_2$$

Hence the complete solution is

$$y = (-\log x + C_1) e^{3x} + \left(-\frac{1}{x} + C_2\right) x e^{3x}$$

Que 1.26. Use variation of parameters method to solve the differential equation $x^2 y'' + xy' - y = x^2 e^x$.

AKTU 2017-18, Marks 07

Answer

$$x^2 y'' + xy' - y = x^2 e^x \quad \dots(1.26.1)$$

$$y'' + \frac{y'}{x} - \frac{y}{x^2} = e^x \quad \dots(1.26.2)$$

Here,

$$R = e^x$$

Consider the equation $y'' + \frac{y'}{x} - \frac{y}{x^2} = 0$ for finding parts of CF

Put $x = e^z$ so that $z = \log x$

So, $[D(D-1) + D - 1]y = 0$

$$(D^2 - 1)y = 0 \quad \dots(1.26.3)$$

Auxiliary equation, $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$\therefore CF = C_1 e^x + C_2 e^{-x} = C_1 x + C_2 \frac{1}{x}$$

Hence parts of CF are x and $\frac{1}{x}$

Let

$$u = x \text{ and } v = \frac{1}{x}$$

Let $y = Ax + \frac{B}{x}$ be the complete solution, where A and B are some suitable functions of x . A and B are determined as follows:

$$\begin{aligned} A &= - \int \frac{Rv}{uv_1 - u_1v} dx + C_1 \\ &= - \int \frac{e^x \frac{1}{x}}{x \left(\frac{-1}{x^2}\right) - 1 \left(\frac{1}{x}\right)} dx + C_1 \end{aligned}$$

$$= - \int \frac{e^x \frac{1}{x}}{\left(\frac{-2}{x}\right)} dx + C_1 = \frac{1}{2} e^x + C_1$$

$$\text{and } B = \int \frac{Ru}{uv_1 - u_1v} dx + C_2 = \int \frac{e^x x}{x \left(\frac{-1}{x^2}\right) - \left(\frac{1}{x}\right)} dx + C_2$$

$$= \int \frac{e^x x}{\left(\frac{-2}{x}\right)} dx + C_2 = -\frac{1}{2} \int x^2 e^x dx + C_2$$

$$= -\frac{1}{2} [x^2 e^x - \int 2xe^x dx] + C_2 = -\frac{1}{2} [x^2 + e^x - 2(x-1)e^x] + C_2$$

$$= -\frac{1}{2} x^2 e^x + (x-1)e^x + C_2$$

Hence the complete solution is given by

$$y = Ax + \frac{B}{x} = \left(\frac{1}{2} e^x + C_1\right)x + \left[-\frac{1}{2} x^2 e^x + (x-1)e^x + C_2\right] \frac{1}{x}$$

$$y = C_1 x + \frac{C_2}{x} + \left(1 - \frac{1}{x}\right)e^x$$

PART-5

Cauchy-Euler Equation.

CONCEPT OUTLINE

Cauchy-Euler Equation: An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = Q$$

Where a_i 's are constants and Q is a function of x , called Cauchy's homogeneous linear equation. Such equations can be reduced to linear differential equations with constant coefficients by the substitution

$$x = e^z \text{ or } z = \log x$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.27. Solve : $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = (\log x) \sin (\log x)$.

AKTU 2014-15, Marks 10

Answer

Since given equation is homogeneous,

Put $x = e^z, \Rightarrow \log x = z$

Also, $x \frac{dy}{dx} = Dy, x^2 \frac{d^2 y}{dx^2} = D(D-1)y, D = \frac{d}{dz}$

The given equation becomes,

$$D(D-1)y + Dy + y = z \sin z$$

$$(D^2 - D + D + 1)y = z \sin z$$

$$(D^2 + 1)y = z \sin z$$

Auxiliary equation is, $m^2 + 1 = 0, \Rightarrow m = \pm i$

$$CF = C_1 \cos z + C_2 \sin z$$

$$PI = \frac{1}{D^2 + 1} z \sin z$$

$$= \text{Imaginary part of } \frac{1}{D^2 + 1} z e^{iz}$$

$$= \text{Imaginary part of } e^{iz} \frac{1}{(D+i)^2 + 1} z$$

$$= \text{Imaginary part of } e^{iz} \frac{1}{D^2 + 2iD - 1 + 1} z$$

$$= \text{Imaginary part of } e^{iz} \frac{1}{D^2 + 2iD} z$$

$$= \text{Imaginary part of } e^{iz} \frac{1}{2iD} \left(\frac{1}{1 + \frac{D}{2i}} \right) z$$

$$= \text{Imaginary part of } e^{iz} \frac{1}{2iD} \left(1 - \frac{D}{2i} \right) z$$

$$= \text{Imaginary part of } e^{iz} \frac{1}{2iD} \left(z - \frac{1}{2i} \right)$$

$$= \text{Imaginary part of } e^{iz} \frac{1}{2i} \left(\frac{z^2}{2} - \frac{z}{2i} \right)$$

$$= \text{Imaginary part of } \frac{1}{2i} (\cos z + i \sin z) \left(\frac{z^2}{2} - \frac{z}{2i} \right)$$

$$= \text{Imaginary part of } (\cos z + i \sin z) \left(\frac{z^2}{4i} + \frac{z}{4} \right)$$

$$= -\frac{z^2}{4} \cos z + \frac{z}{4} \sin z$$

Complete solution, $y = CF + PI$

$$y = C_1 \cos z + C_2 \sin z - \frac{z^2}{4} \cos z + \frac{z}{4} \sin z$$

$$y = C_1 \cos (\log x) + C_2 \sin (\log x) - \frac{1}{4} (\log x)^2 \cos (\log x) + \frac{1}{4} (\log x) \sin (\log x)$$

PART-6

Application of Differential Equations in Solving Engineering Problems.

CONCEPT OUTLINE

Application of differential equation : It is used to calculate the movement or flow of electricity and the motion of an object.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.28. A body of mass m , falling from rest, is subject to the force of gravity and an air resistance proportional to the square of the velocity (i.e., kv^2). If it falls through a distance x and possesses a velocity v at that instant. Prove that $\frac{2kx}{m} = \log \frac{a^2}{a^2 - v^2}$, where $mg = ka^2$.

Answer

The forces acting on the body are :

1. Its weight mg acting vertically downwards.
2. The resistance kv^2 of the air acting vertically upwards.

Accelerating force on the body $= mg - kv^2 = ka^2 - kv^2 = k(a^2 - v^2)$

$$[\because mg = ka^2]$$

By Newton's second law, the equation of motion of the body is

$$mv \frac{dv}{dx} = k(a^2 - v^2)$$

$$\text{or} \quad \frac{v}{a^2 - v^2} dv = \frac{k}{m} dx$$

Integrating,

$$-\frac{1}{2} \log(a^2 - v^2) = \frac{k}{m} x + c \quad \dots(1.28.1)$$

$$\text{Initially, when } x = 0, v = 0 \therefore c = -\frac{1}{2} \log a^2$$

$$\therefore \text{From (1.28.1), we have } -\frac{1}{2} \log(a^2 - v^2) = \frac{k}{m} x - \frac{1}{2} \log a^2$$

$$\text{or} \quad \frac{2kx}{m} = \log a^2 - \log(a^2 - v^2)$$

$$\text{or} \quad \frac{2kx}{m} = \log \frac{a^2}{a^2 - v^2}$$

Que 1.29. A particle of unit mass falls under gravity in a resisting medium whose resistance varies with velocity. Find the relation between distance and velocity if initially the particle starts from rest.

Answer

Let v be the velocity when the particle has fallen distance s in time t from rest so that the resistance due to velocity is kv per unit of mass. Hence the force of resistance on the particle of mass m is mkv against the direction of motion. Also, the force due to gravity, mg is acting vertically downwards.

Equation of motion of the particle is

$$m \frac{dv}{dt} = mg - mkv$$

$$\therefore \frac{dv}{dt} = g - kv$$

$$\frac{dv}{g - kv} = dt$$

Integration gives,

$$-\frac{1}{k} \log(g - kv) = t + c_1 \quad \dots(1.29.1)$$

Since $v = 0$, when $t = 0$

$$\therefore c_1 = -\frac{1}{k} \log g$$

$$\Rightarrow \frac{1}{k} \log \frac{g}{g - kv} = t$$

$$\Rightarrow v = \frac{ds}{dt} = \frac{g}{k} (1 - e^{-kt}) \quad \dots(1.29.2)$$

Integrating (1.29.2), we get

$$s = \frac{g}{k} t + \frac{g}{k^2} e^{-kt} + c_2$$

Since $s = 0$, when $t = 0$,

$$c_2 = -\frac{g}{k^2}$$

$$\therefore s = \frac{g}{k} t + \frac{g}{k^2} (e^{-kt} - 1) \quad \dots(1.29.3)$$

Eliminating t between (1.29.2) and (1.29.3) we get,

$$s = \frac{g}{k^2} \log \left(\frac{g}{g - kv} \right) - \frac{v}{k}$$

which gives a relation between distance and velocity.

Que 1.30. An inductance L of 5 H and a resistance R of 50Ω are connected in series with an emf E volt. If the current i is zero when $t = 0$, find the current i at the end of 0.03 second if $E = 220 \text{ V}$, using the following differential equation

$$L \frac{di}{dt} + Ri = E$$

Answer

The given equation is

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \quad \dots(1.30.1)$$

$$\text{I.F.} = e^{\int R/L dt} = e^{\frac{R}{L}t}$$

Solution of (1.30.1) is

$$ie^{\frac{R}{L}t} = \frac{E}{L} \int e^{\frac{R}{L}t} dt = \frac{E}{R} e^{\frac{R}{L}t} + c_1 \quad \dots(1.30.2)$$

at $t = 0, i = 0$ $\therefore$ From (1.30.2),

$$c_1 = -\frac{E}{R}$$

So, from (1.30.2), $i = \frac{E}{R} \left(1 - e^{-\frac{R}{L}t}\right)$

Putting the values of E, R, L and t , we get

$$i = \frac{220}{50} [1 - e^{-0.2}]$$

$$= 0.80 \text{ A}$$

Que 1.31. The equations of electromotive force in terms of current i for an electrical circuit having resistance R and a condenser of capacity C , in series, is

$$E = Ri + \int \frac{i}{C} dt$$

Find the current i at any time t , when $E = E_0 \sin \omega t$.

Answer

The given equation is

$$Ri + \int \frac{i}{C} dt = E_0 \sin \omega t \quad \dots(1.31.1)$$

Differentiating both sides w.r.t t , we get

$$R \frac{di}{dt} + \frac{i}{C} = \omega E_0 \cos \omega t$$

$$\Rightarrow \frac{di}{dt} + \frac{i}{RC} = \frac{\omega E_0}{R} \cos \omega t \quad \dots(1.31.2)$$

$$\text{I.F.} = e^{\int \frac{1}{RC} dt} = e^{t/RC}$$

Solution of eq. (1.31.2) is

$$ie^{t/RC} = \int \frac{\omega E_0}{R} \cos \omega t e^{t/RC} dt + k$$

$$= \frac{\omega E_0}{R} \frac{e^{t/RC}}{\left(\frac{1}{R^2 C^2} + \omega^2\right)} \left(\frac{1}{RC} \cos \omega t + \omega \sin \omega t\right) + k$$

$$\Rightarrow ie^{t/RC} = \frac{\omega E_0 RC^2}{(1 + R^2 C^2 \omega^2)} \left(\frac{1}{RC} \cos \omega t + \omega \sin \omega t\right) e^{t/RC} + k$$

$$\Rightarrow i = \frac{\omega E_0 C}{1 + R^2 C^2 \omega^2} (\cos \omega t + RC \omega \sin \omega t) + k e^{-t/RC}$$

where k is an arbitrary constant of integration.



2 UNIT

Laplace Transform

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Existence Theorem
- Part-2 :** Properties of Laplace Transform 2-6F to 2-8F
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Differential Equations

2-1 F (Sem-2)

2-2 F (Sem-2)

Laplace Transform

PART-1

Laplace Transform, Existence Theorem.

CONCEPT OUTLINE

Introduction : Laplace transforms help in solving the differential equations with boundary values without finding the general solution and the values of the arbitrary constants.

- **Bilateral Laplace Transform :**

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$$

- **Unilateral Laplace Transform :**

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.1. What is Laplace transform ? Also write existence theorem.

Answer

Laplace transform :

Let $f(t)$ be function defined for all positive values of t ($t \geq 0$), then

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

provided the integral exists, is called the Laplace transform of $f(t)$. It is denoted as

$$L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

Existence theorem :

If $f(t)$ is continuous for $t > 0$ and is of exponential order α , then

$$L[f(t)] = F(s) \text{ exists for } s > \alpha.$$

In other words, if $f(t)$ is a function of class A, $L[f(t)]$ exists.

$$\int_0^{\infty} e^{-st} f(t) dt = \int_0^{t_0} e^{-st} f(t) dt + \int_{t_0}^{\infty} e^{-st} f(t) dt = I_1 + I_2$$

I_1 exists since $f(t)$ is continuous in the interval $[0, t_0]$

$$\begin{aligned}
 |I_2| &\leq \int_0^\infty |e^{-st} f(t)| dt \leq \int_0^\infty e^{-at} |f(t)| dt \\
 &\leq \int_0^\infty e^{-at} M e^{at} dt \\
 &\leq \int_0^\infty e^{-i(s-a)t} M dt = \frac{M}{s-a}
 \end{aligned}$$

Thus, the Laplace transform exists for $s > a$.

Que 2.2. Find Laplace transform of following functions :

- | | |
|--------------------|-------------------|
| i. $L(1)$ | ii. $L(t^n)$ |
| iii. $L(e^{at})$ | iv. $L(\cos at)$ |
| v. $L(\sin at)$ | vi. $L(\cosh at)$ |
| vii. $L(\sinh at)$ | |

Answer

i. $L(1) = \int_0^\infty 1 \cdot e^{-st} dt = \left[-\frac{e^{-st}}{s} \right]_0^\infty$

$$= -\frac{1}{s} \left[\frac{1}{e^{st}} \right]_0^\infty = -\frac{1}{s} [0 - 1] = \frac{1}{s}$$

ii. $L(t^n) = \int_0^\infty e^{-st} t^n dt$

Putting, $st = x$ or $t = \frac{x}{s}$ or $dt = \frac{dx}{s}$

Thus, we have, $L(t^n) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx$

$\left[\int_0^\infty e^{-x} x^n dx = n! \right]$

$$L(t^n) = \frac{n!}{s^{n+1}}$$

iii. $L(e^{at}) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt$

$$= \int_0^\infty e^{-(s-a)t} dt = \int_0^\infty e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty = -\frac{1}{s-a} \left[\frac{1}{e^{(s-a)t}} \right]_0^\infty$$

$$= \frac{-1}{(s-a)} (0 - 1) = \frac{1}{s-a}$$

iv. $L(\cos at) = L\left(\frac{e^{iat} + e^{-iat}}{2}\right) \left[\because \cos at = \frac{e^{iat} + e^{-iat}}{2} \right]$

$$= \frac{1}{2} [L(e^{iat}) + L(e^{-iat})] = \frac{1}{2} [L(e^{iat}) + L(e^{-iat})]$$

$$= \frac{1}{2} \left[\frac{1}{s-ia} + \frac{1}{s+ia} \right] = \frac{1}{2} \left[\frac{s+ia + s-ia}{s^2 + a^2} \right]$$

$$= \frac{s}{s^2 + a^2}$$

v. $L(\sin at) = L\left(\frac{e^{iat} - e^{-iat}}{2i}\right) \left[\because \sin at = \frac{e^{iat} - e^{-iat}}{2i} \right]$

$$= \frac{1}{2i} [L(e^{iat}) - L(e^{-iat})] = \frac{1}{2i} [L(e^{iat}) - L(e^{-iat})]$$

$$= \frac{1}{2i} \left[\frac{1}{s-ia} - \frac{1}{s+ia} \right] = \frac{1}{2i} \left[\frac{s+ia - s+ia}{s^2 + a^2} \right]$$

$$= \frac{1}{2i} \left[\frac{2ia}{s^2 + a^2} \right] = \frac{a}{s^2 + a^2}$$

vi. $L(\cosh at) = L\left(\frac{e^{at} + e^{-at}}{2}\right) \left[\because \cosh at = \frac{e^{at} + e^{-at}}{2} \right]$

$$= \frac{1}{2} L(e^{at}) + \frac{1}{2} L(e^{-at})$$

$$= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right] \left[\because L(e^{at}) = \frac{1}{s-a} \right]$$

$$= \frac{1}{2} \left[\frac{s+a + s-a}{s^2 - a^2} \right] = \frac{s}{s^2 - a^2}$$

vii. $L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2}\right) \left[\because \sinh at = \frac{e^{at} - e^{-at}}{2} \right]$

$$= \frac{1}{2} [L(e^{at}) - L(e^{-at})] = \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[\frac{s+a - s+a}{s^2 - a^2} \right]$$

$$= \frac{a}{s^2 - a^2}$$

Que 2.3. Find the Laplace transform of $t^2 \cos at$.

Answer

$$\begin{aligned}
 L(t^2 \cos at) &= L\left(t^2 \frac{e^{iat} + e^{-iat}}{2}\right) = \frac{1}{2} [L(t^2 e^{iat}) + L(t^2 e^{-iat})] \\
 &= \frac{1}{2} \left[\frac{2!}{(s-ia)^3} + \frac{2!}{(s+ia)^3} \right] = \frac{(s+ia)^3 + (s-ia)^3}{(s+ia)^3 (s-ia)^3} \\
 &= \frac{(s^3 + 3ias^2 - 3a^2s - ia^3) + (s^3 - 3ias^2 - 3a^2s + ia^3)}{(s^2 + a^2)^3} \\
 &= \frac{2s^3 - 6a^2s}{(s^2 + a^2)^3} = \frac{2s(s^2 - 3a^2)}{(s^2 + a^2)^3}
 \end{aligned}$$

Que 2.4. Find the Laplace transform of $t \sin at$.**Answer**

$$\begin{aligned}
 L(t \sin at) &= L\left(t \frac{e^{iat} - e^{-iat}}{2i}\right) = \frac{1}{2i} [L(te^{iat}) - L(te^{-iat})] \\
 &= \frac{1}{2i} \left[\frac{1}{(s-ia)^2} - \frac{1}{(s+ia)^2} \right] = \frac{1}{2i} \left[\frac{(s+ia)^2 - (s-ia)^2}{(s-ia)^2 (s+ia)^2} \right] \\
 &= \frac{1}{2i} \left[\frac{(s^2 + 2ias - a^2) - (s^2 - 2ias - a^2)}{(s^2 + a^2)^2} \right] \\
 &= \frac{1}{2i} \left[\frac{4ias}{(s^2 + a^2)^2} \right] = \frac{2as}{(s^2 + a^2)^2}
 \end{aligned}$$

Que 2.5. Find the Laplace transform of

$$f(t) = \begin{cases} t^2, & 0 < t < 2 \\ t-1, & 2 < t < 3 \\ 7 & t > 3 \end{cases}$$

Answer

$$\begin{aligned}
 L(f(t)) &= \int_0^\infty e^{-st} f(t) dt \\
 &= \int_0^2 t^2 e^{-st} dt + \int_2^3 (t-1)e^{-st} dt + \int_3^\infty 7e^{-st} dt
 \end{aligned}$$

$$\begin{aligned}
 &= \left[t^2 \left(\frac{e^{-st}}{(-s)} \right) - 2t \frac{e^{-st}}{(-s)^2} + \frac{e^{-st}}{(-s)^3} \right]_0^2 \\
 &\quad + \left[(t-1) \left(\frac{e^{-st}}{(-s)} \right) - \frac{e^{-st}}{(-s)^2} \right]_2^3 + 7 \left[\frac{e^{-st}}{(-s)} \right]_3^\infty \\
 &= \left[-4 \left(\frac{e^{-2s}}{s} \right) - 4 \left(\frac{e^{-2s}}{s^2} \right) - 2 \left(\frac{e^{-2s}}{s^3} \right) + \frac{2}{s^3} \right] \\
 &\quad + \left[2 \left(\frac{e^{-3s}}{-s} \right) - \frac{e^{-3s}}{s^2} + \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2} \right] + 7 \left(0 + \frac{e^{-3s}}{s} \right) \\
 &= \frac{2}{s^3} + e^{-2s} \left[-\frac{4}{s} - \frac{4}{s^2} - \frac{2}{s^3} \right] \\
 &\quad + e^{-3s} \left[-\frac{2}{s} - \frac{1}{s^2} \right] + e^{-2s} \left[\frac{1}{s} + \frac{1}{s^2} \right] + e^{-3s} \left[\frac{7}{s} \right] \\
 &= \frac{2}{s^3} + e^{-2s} \left[-\frac{4}{s} - \frac{4}{s^2} - \frac{2}{s^3} + \frac{1}{s} + \frac{1}{s^2} \right] \\
 &\quad + e^{-3s} \left[-\frac{2}{s} - \frac{1}{s^2} + \frac{7}{s} \right] \\
 &= \frac{2}{s^3} + e^{-2s} \left[-\frac{3}{s} - \frac{3}{s^2} - \frac{2}{s^3} \right] + e^{-3s} \left[\frac{5}{s} - \frac{1}{s^2} \right] \\
 &= \frac{2}{s^3} - \frac{e^{-2s}}{s^3} (2 + 3s + 3s^2) + \frac{e^{-3s}}{s^2} (5s - 1)
 \end{aligned}$$

PART-2

Properties of Laplace Transform.

Questions-Answers**Long Answer Type and Medium Answer Type Questions****Que 2.6.** Write the properties of Laplace transform.**Answer****Properties of Laplace transform :**

- Linearity property :** If a, b, c be any constants and f, g, h any functions of t , then

$$L[af(t) + bg(t) - ch(t)] = aL\{f(t)\} + bL\{g(t)\} - cL\{h(t)\}$$

Proof:

$$\begin{aligned} \text{L.H.S} &= \int_0^\infty e^{-st} [af(t) + bg(t) - ch(t)] dt \\ &= a \int_0^\infty e^{-st} f(t) dt + b \int_0^\infty e^{-st} g(t) dt - c \int_0^\infty e^{-st} h(t) dt \\ &= aL\{f(t)\} + bL\{g(t)\} - cL\{h(t)\} \end{aligned}$$

2. **First shifting property:** If $L\{f(t)\} = F(s)$, then

$$L\{e^{at} f(t)\} = F(s - a)$$

Proof:

$$\begin{aligned} L\{e^{at} f(t)\} &= \int_0^\infty e^{-st} e^{at} f(t) dt = \int_0^\infty e^{-(s-a)t} f(t) dt \\ &= \int_0^\infty e^{-rt} f(t) dt \quad (\text{where, } r = s - a) \\ &= F(r) = F(s - a) \end{aligned}$$

3. **Change of scale property:** If $L\{f(t)\} = F(s)$,

$$\text{then } L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$\begin{aligned} \text{Proof: } L\{f(at)\} &= \int_0^\infty e^{-st} f(at) dt = \int_0^\infty e^{-\frac{s}{a}u} f(u) \frac{du}{a} \quad \left[\begin{array}{l} \text{Put } at = u \\ dt = du/a \end{array} \right] \\ &= \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u) du = \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)t} f(t) dt \\ &= \frac{1}{a} F\left(\frac{s}{a}\right) \end{aligned}$$

Que 2.7. Find the Laplace transform of

i. $e^{-3t} (2 \cos 5t - 3 \sin 5t)$

ii. $e^{2t} \cos^2 t$

iii. $\sqrt{t} e^{3t}$

Answer

$$\begin{aligned} \text{i. } L\{e^{-3t} (2 \cos 5t - 3 \sin 5t)\} &= 2L\{e^{-3t} \cos 5t\} - 3L\{e^{-3t} \sin 5t\} \\ &= 2 \frac{s+3}{(s+3)^2 + 5^2} - 3 \frac{5}{(s+3)^2 + 5^2} = \frac{2s-9}{s^2 + 6s + 34} \end{aligned}$$

$$\text{ii. Since } L(\cos^2 t) = \frac{1}{2} L(1 + \cos 2t) = \frac{1}{2} \left\{ \frac{1}{s} + \frac{s}{s^2 + 4} \right\}$$

∴ By shifting property, we get

$$L\{e^{2t} \cos^2 t\} = \frac{1}{2} \left\{ \frac{1}{s-2} + \frac{s-2}{(s-2)^2 + 4} \right\}$$

$$\text{iii. Since } L(\sqrt{t}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{(1/2)\Gamma\pi}{s^{3/2}}$$

∴ By shifting property, we obtain

$$L\{e^{3t} \sqrt{t}\} = \frac{\sqrt{\pi}}{2} \frac{1}{(s-3)^{3/2}}$$

Que 2.8. Find $L\left\{\frac{\sin at}{t}\right\}$, given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left\{\frac{1}{s}\right\}$.

Answer

By the change of scale property,

$$\begin{aligned} L\left\{\frac{\sin at}{at}\right\} &= \frac{1}{a} \tan^{-1}\left\{\frac{1}{s/a}\right\} \\ &= \frac{1}{a} \tan^{-1}\left\{\frac{a}{s}\right\} \text{ i.e., } L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left\{\frac{a}{s}\right\} \end{aligned}$$

Que 2.9. If $L(\cos^2 t) = \frac{s^2 + 2}{s(s^2 + 4)}$, find $L(\cos^2 at)$.

Answer

$$\text{We have } L(\cos^2 t) = \frac{s^2 + 2}{s(s^2 + 4)}$$

By change of scale property, we have

$$\begin{aligned} L(\cos^2 at) &= \frac{1}{a} \cdot \frac{\left(\frac{s}{a}\right)^2 + 2}{\frac{s}{a} \left[\left(\frac{s}{a}\right)^2 + 4\right]} = \frac{1}{a} \left[\frac{s^2 + 2a^2}{s(s^2 + 4a^2)} \right] \\ &= \frac{s^2 + 2a^2}{s(s^2 + 4a^2)} \end{aligned}$$

PART-3

Laplace Transform of Derivates and Integrals.

CONCEPT OUTLINE**Laplace transform of $t \cdot f(t)$ (multiplication by t):** $L[f(t) = F(s)], \text{ then}$

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)]$$

Laplace transform of $\frac{1}{t} f(t)$ (Division by t):

If $L[f(t)] = F(s)$, then $\left[\frac{1}{t} f(t) \right] = \int_s^\infty F(s) ds$

Questions-Answers**Long Answer Type and Medium Answer Type Questions****Que 2.10.** Determine the Laplace transform of the derivative.**Answer****Laplace transform of the derivative of $f(t)$:**

$$L[f'(t)] = s L[f(t)] - f(0) \quad [\text{where, } L[f(t)] = F(s)]$$

$$\text{Proof: } L[f'(t)] = \int_0^\infty e^{-st} f'(t) dt$$

Integrating by parts, we get

$$\begin{aligned} L[f'(t)] &= \left[e^{-st} f(t) \right]_0^\infty - \int_0^\infty (-se^{-st}) f(t) dt \\ &= -f(0) + s \int_0^\infty e^{-st} f(t) dt \quad (e^{-st} f(t) = 0, \text{ when } t = \infty) \\ &= -f(0) + s L[f(t)] \end{aligned}$$

$$\Rightarrow L[f'(t)] = s L[f(t)] - f(0).$$

Laplace transform of derivative of order n :

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \dots - f^{n-1}(0).$$

Proof: As we know,

$$L[f'(t)] = s L[f(t)] - f(0) \quad \dots(2.10.1)$$

Replacing $f(t)$ by $f'(t)$ and $f'(t)$ by $f''(t)$ in (2.10.1), we get

$$L[f''(t)] = s L[f'(t)] - f'(0) \quad \dots(2.10.2)$$

Putting the value of $L[f'(t)]$ from eq. (2.10.1) in (2.10.2), we have

$$L[f''(t)] = s [s L[f(t)] - f(0)] - f'(0)$$

$$\Rightarrow L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0)$$

$$\text{Similarly, } L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - s f'(0) - f''(0)$$

$$L[f^{iv}(t)] = s^4 L[f(t)] - s^3 f(0) - s^2 f'(0) - s f''(0) - f'''(0)$$

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \dots - f^{n-1}(0)$$

Que 2.11. Determine the Laplace transform of Integrals.**Answer****Laplace transform of Integrals of $f(t)$:**

$$L\left[\int_0^t f(t) dt\right] = \frac{1}{s} F(s), \quad [\text{where, } L[f(t)] = F(s)]$$

Proof:

$$\text{Let } \phi(t) = \int_0^t f(t) dt \text{ and } \phi(0) = 0 \text{ then } \phi'(t) = f(t)$$

As we know,

$$L[\phi'(t)] = s L[\phi(t)] - \phi(0)$$

$$\Rightarrow L[\phi'(t)] = s L[\phi(t)] \quad [\phi(0) = 0]$$

$$\Rightarrow L[\phi(t)] = \frac{1}{s} L[\phi'(t)]$$

Putting the value of $\phi(t)$ and $\phi'(t)$, we get

$$L\left[\int_0^t f(t) dt\right] = \frac{1}{s} L[f(t)] \Rightarrow L\left[\int_0^t f(t) dt\right] = \frac{1}{s} F(s)$$

Que 2.12. Find the Laplace transform of $t^n f(t)$ (multiplication by t)**Answer** $L[f(t) = F(s)], \text{ then}$

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)]$$

Proof:

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt \quad \dots(2.12.1)$$

Differentiating (2.12.1) wrt 's' we get

$$\therefore \frac{d}{ds} [F(s)] = \frac{d}{ds} \left[\int_0^\infty e^{-st} f(t) dt \right] = \int_0^\infty \frac{d}{ds} (e^{-st}) f(t) dt$$

$$= \int_0^{\infty} (-t e^{-st}) f(t) dt = \int_0^{\infty} e^{-st} [-t f(t)] dt$$

$$= L[-t f(t)] \text{ or } L[t f(t)] = (-1)^1 \frac{d}{ds} [F(s)]$$

Similarly, $L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} [F(s)]$

$$L[t^3 f(t)] = (-1)^3 \frac{d^3}{ds^3} [F(s)]$$

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)]$$

Que 2.13. Find the Laplace transform of

i. $t \cos at$

ii. $t^2 \sin at$

iii. $t^3 e^{-3t}$

iv. $te^{-t} \sin 3t$

Answer

i. Since

$$L(\cos at) = s/(s^2 + a^2)$$

$$\therefore L(t \cos at) = -\frac{d}{ds} \left(\frac{s}{s^2 + a^2} \right) = -\frac{s^2 + a^2 - s(2s)}{(s^2 + a^2)^2}$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

ii. Since

$$L(\sin at) = \frac{a}{s^2 + a^2}$$

$$\therefore L(t^2 \sin at) = (-1)^2 \frac{d^2}{ds^2} \left(\frac{a}{s^2 + a^2} \right)$$

$$= \frac{d}{ds} \left\{ \frac{-2as}{(s^2 + a^2)^2} \right\} = \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}$$

iii. Since

$$L(e^{-3t}) = 1/(s + 3),$$

$$\therefore L(t^3 e^{-3t}) = (-1)^3 \frac{d^3}{ds^3} \left(\frac{1}{s + 3} \right) = -\frac{(-1)^3 \cdot 3!}{(s + 3)^{3+1}} = 6/(s + 3)^4.$$

iv. Since

$$L(\sin 3t) = \frac{3}{s^2 + 3^2},$$

$$\text{Therefore, } L(t \sin 3t) = -\frac{d}{ds} \left(\frac{3}{s^2 + 3^2} \right) = \frac{6s}{(s^2 + 9)^2}$$

Now using the shifting property, we get

$$L(e^t t \sin 3t) = \frac{6(s+1)}{[(s+1)^2 + 9]^2} = \frac{6(s+1)}{(s^2 + 2s + 10)^2}$$

Que 2.14. Find the Laplace transform of $t^2 \cos at$.

Answer

Since, $L(\cos at) = \frac{s}{s^2 + a^2}$

$$L(t^2 \cos at) = (-1)^2 \frac{d^2}{ds^2} \left[\frac{s}{s^2 + a^2} \right] = \frac{d}{ds} \left[\frac{(s^2 + a^2) - s(2s)}{(s^2 + a^2)^2} \right]$$

$$= \frac{d}{ds} \left[\frac{a^2 - s^2}{(s^2 + a^2)^2} \right]$$

$$= \frac{(s^2 + a^2)^2 (-2s) - (a^2 - s^2) 2(s^2 + a^2)(2s)}{(s^2 + a^2)^4}$$

$$= \frac{-2s^3 - 2a^2 s - 4a^2 s + 4s^3}{(s^2 + a^2)^3}$$

$$= \frac{2s(s^2 - 3a^2)}{(s^2 + a^2)^3}$$

Que 2.15. Obtain the Laplace transform of $t^3 e^t \sin 4t$.

Answer

$$L(\sin 4t) = \frac{4}{s^2 + 16}, \quad L(e^t \sin 4t) = \frac{4}{(s-1)^2 + 16}$$

$$L(t e^t \sin 4t) = -\frac{d}{ds} \left(\frac{4}{s^2 - 2s + 17} \right) = \frac{4(2s-2)}{(s^2 - 2s + 17)^2}$$

$$L(t^2 e^t \sin 4t) = -\frac{d}{ds} \left(\frac{4(2s-2)}{(s^2 - 2s + 17)^2} \right)$$

$$= -4 \frac{(s^2 - 2s + 17)^2 \times 2 - (2s-2) \times 2(s^2 - 2s + 17)(2s-2)}{(s^2 - 2s + 17)^4}$$

$$= \frac{-4(2s^3 - 4s + 34 - 8s^2 + 16s - 8)}{(s^2 - 2s + 17)^3}$$

$$= \frac{-4(-6s^2 + 12s + 26)}{(s^2 - 2s + 17)^3} = \frac{8(3s^2 - 6s - 13)}{(s^2 - 2s + 17)^3}$$

Que 2.16. Find the Laplace transform of $f(t) = \int_0^t \frac{\sin at}{t} dt$.

Answer

$$L(\sin at) = \frac{a}{s^2 + a^2}$$

$$L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{s^2 + a^2} ds = \left[\tan^{-1}\left(\frac{s}{a}\right) \right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$

$$\text{Hence, } L\left[\int_0^t \frac{\sin at}{t} dt\right] = \frac{1}{s} \cot^{-1}\left(\frac{s}{a}\right)$$

Que 2.17. If $f(t) = \frac{e^{at} - \cos bt}{t}$, find the Laplace transform of $f(t)$.

Answer

$$f(t) = \frac{e^{at} - \cos bt}{t}$$

$$= \frac{e^{at}}{t} - \frac{\cos bt}{t}$$

We know that, $L(e^{at} - \cos bt) = \left(\frac{1}{s-a} - \frac{s}{s^2 + b^2} \right)$

$$L\left(\frac{e^{at} - \cos bt}{t}\right) = \int_s^\infty \left(\frac{1}{s-a} - \frac{s}{s^2 + b^2} \right) ds$$

$$= \left[\log(s-a) - \frac{1}{2} \log(s^2 + b^2) \right]_s^\infty$$

$$= \left[\frac{2 \log(s-a) - \log(s^2 + b^2)}{2} \right]_s^\infty$$

$$= \frac{1}{2} [\log(s-a)^2 - \log(s^2 + b^2)]_s^\infty$$

$$= \frac{1}{2} \left[\log \frac{(s-a)^2}{s^2 + b^2} \right]_s^\infty = \frac{1}{2} \left[\log \left(\frac{\left(1 - \frac{a}{s}\right)^2}{\left(1 + \frac{b^2}{s^2}\right)} \right) \right]_s^\infty$$

$$= \frac{1}{2} \left[0 - \log \left(\frac{\left(1 - \frac{a}{s}\right)^2}{\left(1 + \frac{b^2}{s^2}\right)} \right) \right] = \frac{1}{2} \left[\log \frac{(s^2 + b^2)}{(s-a)^2} \right]$$

Que 2.18. Find the Laplace transform of $\frac{\sin 2t}{t}$.

Answer

$$L(\sin 2t) = \frac{2}{s^2 + 4}$$

$$L\left(\frac{\sin 2t}{t}\right) = \int_s^\infty \frac{2}{s^2 + 4} ds = 2 \left[\frac{1}{2} \tan^{-1}\left(\frac{s}{2}\right) \right]_s^\infty$$

$$= \left[\tan^{-1} \infty - \tan^{-1}\left(\frac{s}{2}\right) \right] = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \cot^{-1}\left(\frac{s}{2}\right)$$

PART-4

Unit Step Function, Laplace transform of Periodic Function.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.19. Define unit step function.

OR

Express the following function in term of unit step functions and find its Laplace transform :

$$f(t) = \begin{cases} 8, & t < 2 \\ 6, & t \geq 2 \end{cases}$$

Answer

Unit step function :

The unit step functions $u(t-a)$ is defined as follows :

$$u(t-a) = \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t \geq a \end{cases} \quad (\text{where, } a \geq 0)$$



Fig. 2.19.1.

Numerical :

$$\begin{aligned}
 f(t) &= \begin{cases} 8+0 & t < 2 \\ 8-2 & t \geq 2 \end{cases} \\
 &= 8 + \begin{cases} 0 & t < 2 \\ -2 & t \geq 2 \end{cases} \\
 &= 8 + (-2) \begin{cases} 0, & t < 2 \\ 1, & t \geq 2 \end{cases} \\
 &= 8 - 2u(t-2)
 \end{aligned}$$

$$L[f(t)] = 8L(1) - 2L[u(t-2)] = \frac{8}{s} - 2 \frac{e^{-2s}}{s}$$

Que 2.20. Determine the Laplace transform of periodic functions.**Answer**Let $f(t)$ be a periodic function with period T , then

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$$

Proof:

$$\begin{aligned}
 L[f(t)] &= \int_0^\infty e^{-st} f(t) dt \\
 &= \int_0^T e^{-st} f(t) dt + \int_T^{2T} e^{-st} f(t) dt + \int_{2T}^{3T} e^{-st} f(t) dt + \dots
 \end{aligned}$$

Substituting

 $t = u + T$ in second integral and $t = u + 2T$ in third integral, and so on

$$\begin{aligned}
 L[f(t)] &= \int_0^T e^{-st} f(t) dt + \int_0^T e^{-s(u+T)} f(u+T) du + \int_0^T e^{-s(u+2T)} f(u+2T) du + \dots \\
 &= \int_0^T e^{-st} f(t) dt + e^{-sT} \int_0^T e^{-su} f(u) du + e^{-2sT} \int_0^T e^{-su} f(u) du + \dots \\
 &\quad [f(u) = f(u+T) = f(u+2T) = f(u+3T) = \dots]
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^T e^{-st} f(t) dt + e^{-sT} \int_0^T e^{-st} f(t) dt + e^{-2sT} \int_0^T e^{-st} f(t) dt + \dots \\
 &= [1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \dots] \int_0^T e^{-st} f(t) dt \left[1 + a + a^2 + a^3 + \dots = \frac{1}{1-a} \right] \\
 &= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt
 \end{aligned}$$

Que 2.21. Find the Laplace transform of the waveform

$$f(t) = \left(\frac{2t}{3} \right), \quad 0 \leq t \leq 3.$$

Answer

$$\begin{aligned}
 L[f(t)] &= \frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}} \\
 L\left[\frac{2t}{3}\right] &= \frac{1}{1 - e^{-3s}} \int_0^3 e^{-st} \left(\frac{2t}{3}\right) dt \\
 &= \frac{1}{1 - e^{-3s}} \cdot \frac{2}{3} \left[\frac{te^{-st}}{-s} - (1) \frac{e^{-st}}{s^2} \right]_0^3 \\
 &= \frac{2}{3} \frac{1}{1 - e^{-3s}} \left[\frac{3e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} \right] \\
 &= \frac{2}{3} \frac{1}{1 - e^{-3s}} \left[\frac{3e^{-3s}}{-s} + \frac{1 - e^{-3s}}{s^2} \right] \\
 &= \frac{2e^{-3s}}{-s(1 - e^{-3s})} + \frac{2}{3s^2}
 \end{aligned}$$

Que 2.22. Find the Laplace transform of the function (Half wave rectifier).

$$f(t) = \begin{cases} \sin \omega t & \text{for } 0 < t < \frac{\pi}{\omega} \\ 0 & \text{for } \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

Answer

$$\begin{aligned}
 L[f(t)] &= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt \\
 &= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \int_0^{\frac{2\pi}{\omega}} e^{-st} f(t) dt \quad \left[\begin{array}{l} f(t) \text{ is periodic function} \\ T = \frac{2\pi}{\omega} \end{array} \right]
 \end{aligned}$$

$$= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} \times 0 \times dt \right]$$

$$= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt$$

We know that

$$\int e^{ax} \sin bx \, dx = e^{ax} \frac{(a \sin bx - b \cos bx)}{a^2 + b^2}$$

$$\therefore L[f(t)] = \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-st} (-s \sin \omega t - \omega \cos \omega t)}{s^2 + \omega^2} \right]_0^{\pi/\omega}$$

$$= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\frac{\omega e^{-\frac{\pi s}{\omega}} + \omega}{s^2 + \omega^2} \right] = \frac{\omega \left[1 + e^{-\frac{\pi s}{\omega}} \right]}{(s^2 + \omega^2) \left[1 - e^{-\frac{2\pi s}{\omega}} \right]}$$

$$= \frac{\omega}{(s^2 + \omega^2) \left[1 - e^{-\frac{\pi s}{\omega}} \right]}$$

PART-5

Inverse Laplace Transform, Convolution Theorem.

CONCEPT OUTLINE

Inverse Laplace transforms : Now we obtain $f(t)$ when $F(s)$ is given, then we say that inverse Laplace transform of $F(s)$ is $f(t)$.

If $L[f(t)] = F(s)$, then $L^{-1}[F(s)] = f(t)$.

where, L^{-1} is called the inverse Laplace transform operator.

Important formulae of inverse Laplace transform :

- | | |
|---|---|
| 1. $L^{-1}\left(\frac{1}{s}\right) = 1$ | 2. $L^{-1}\frac{1}{s^n} = \frac{t^{n-1}}{(n-1)!}$ |
| 3. $L^{-1}\frac{1}{s-a} = e^{at}$ | 4. $L^{-1}\frac{s}{s^2-a^2} = \cosh at$ |
| 5. $L^{-1}\frac{1}{s^2-a^2} = \frac{1}{a} \sinh at$ | 6. $\frac{1}{s^2+a^2} = \frac{1}{a} \sin at$ |
| 7. $L^{-1}\frac{s}{s^2+a^2} = \cos at$ | |

$$8. L^{-1} \frac{s-a}{(s-a)^2 + b^2} = e^{at} \cos bt \quad 9. L^{-1}(1) = s(t)$$

$$10. L^{-1} \left\{ \frac{1}{s} F(s) \right\} = \int_0^t f(u) \, du$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.23. Find the inverse Laplace transform of the following :

- | | | |
|-------------------------|---------------------------|----------------------------|
| i. $\frac{1}{s-2}$ | ii. $\frac{1}{s^2-9}$ | |
| iii. $\frac{s}{s^2-16}$ | iv. $\frac{1}{(s-2)^2+1}$ | v. $\frac{s-1}{(s-1)^2+4}$ |

Answer

- i. $L^{-1} \frac{1}{s-2} = e^{2t}$
- ii. $L^{-1} \frac{1}{s^2-9} = L^{-1} \frac{1}{3} \left[\frac{3}{s^2-(3)^2} \right] = \frac{1}{3} \sinh 3t$
- iii. $L^{-1} \frac{s}{s^2-16} = L^{-1} \left[\frac{s}{s^2-(4)^2} \right] = \cosh 4t$
- iv. $L^{-1} \frac{1}{(s-2)^2+1} = e^{2t} \sin t$
- v. $L^{-1} \frac{s-1}{(s-1)^2+4} = \frac{s-1}{(s-1)^2+(2)^2} = e^t \cos 2t$

Que 2.24. Show that $\frac{1}{s^{1/2}} = L \left[\frac{1}{\sqrt{\pi t}} \right]$.

Answer

We have to show that $\frac{1}{s^{1/2}} = L \left[\frac{1}{\sqrt{\pi t}} \right]$.

$$\text{Now, } L^{-1} \left(\frac{1}{s^n} \right) = \frac{t^{n-1}}{(n-1)!} = \frac{t^{n-1}}{\Gamma(n)}$$

$$\text{So, } L^{-1} \left\{ \frac{1}{s^{1/2}} \right\} = \frac{t^{\frac{1}{2}-1}}{|1/2|} = \frac{t^{-1/2}}{|1/2|} = \frac{t^{-1/2}}{\sqrt{\pi}}$$

$$\Rightarrow L^{-1} \left\{ \frac{1}{s^{1/2}} \right\} = \frac{1}{\sqrt{\pi t}} \Rightarrow \frac{1}{s^{1/2}} = L \left[\frac{1}{\sqrt{\pi t}} \right]$$

Que 2.25. Find the inverse transforms of

i. $\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}$

ii. $\frac{s}{s^2 + 4s + 13}$

Answer

i. Here the denominator = $(s-1)(s-2)(s-3)$.

$$\text{So let, } \frac{2s^2 - 6s + 5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\text{Then } A = [2 \cdot 1^2 - 6 \cdot 1 + 5] / (1-2)(1-3) = \frac{1}{2}$$

$$B = [2 \cdot 2^2 - 6 \cdot 2 + 5] / (2-1)(2-3) = -1$$

$$\text{and } C = [2 \cdot 3^2 - 6 \cdot 3 + 5] / (3-1)(3-2) = \frac{5}{2}$$

$$L^{-1} \left\{ \frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6} \right\} = \frac{1}{2} L^{-1} \left\{ \frac{1}{s-1} \right\} - L^{-1} \left\{ \frac{1}{s-2} \right\} + \frac{5}{2} L^{-1} \left\{ \frac{1}{s-3} \right\}$$

$$= \frac{1}{2} e^t - e^{2t} + \frac{5}{2} e^{3t}$$

ii. $L^{-1} \left\{ \frac{s}{s^2 + 4s + 13} \right\} = L^{-1} \left\{ \frac{s+2-2}{(s+2)^2 + (3)^2} \right\}$

$$= L^{-1} \left\{ \frac{s+2}{(s+2)^2 + (3)^2} \right\} - L^{-1} \left\{ \frac{2}{(s+2)^2 + (3)^2} \right\}$$

$$= e^{-2t} L^{-1} \left\{ \frac{s}{s^2 + 3^2} \right\} - e^{-2t} L^{-1} \left\{ \frac{2}{s^2 + 3^2} \right\}$$

$$= e^{-2t} \cos 3t - \frac{2}{3} e^{-2t} \sin 3t$$

Que 2.26. Find the inverse Laplace transform of $\frac{5s+3}{(s-1)(s^2+2s+5)}$.

Answer

$$L^{-1} \left\{ \frac{5s+3}{(s-1)(s^2+2s+5)} \right\}$$

$$\text{Let } \frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

$$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$$

$$5s+3 = s^2(A+B) + s(2A-B+C) + (5A-C)$$

Comparing the coefficients,

$$A+B=0 \quad \dots(2.26.1)$$

$$2A-B+C=5 \quad \dots(2.26.2)$$

$$5A-C=3 \quad \dots(2.26.3)$$

On adding equations eq. (2.26.1) and (2.26.2), we get

$$3A+C=5 \quad \dots(2.26.4)$$

Adding equations eq. (2.26.3) and (2.26.4), we get

$$8A=8 \Rightarrow A=1$$

Putting $A=1$ in (2.26.1) we get

$$C=2$$

Putting $A=1, C=2$ in (2.26.2), we get

$$B=-1$$

Thus

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{-s+2}{s^2+2s+5} = \frac{1}{s-1} - \frac{s-2}{(s+1)^2+2^2}$$

$$= \frac{1}{s-1} - \frac{s+1}{(s+1)^2+(2)^2} + \frac{3}{(s+1)^2+2^2}$$

$$L^{-1} \left\{ \frac{5s+3}{(s-1)(s^2+2s+5)} \right\} = L^{-1} \left\{ \frac{1}{s-1} \right\} + L^{-1} \left\{ \frac{3}{(s+1)^2+2^2} \right\} - L^{-1} \left\{ \frac{s+1}{(s+1)^2+2^2} \right\}$$

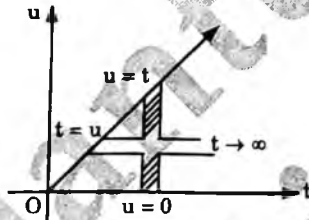
$$= e^t + 3e^{-t} L^{-1} \left\{ \frac{1}{s^2+2^2} \right\} - e^{-t} L^{-1} \left\{ \frac{s}{s^2+2^2} \right\}$$

$$= e^t + 3e^{-t} \frac{1}{2} \sin 2t - e^{-t} \cos 2t$$

Que 2.27. State and prove convolution theorem.

Answer**Statement :**If $L^{-1}\{F(s)\} = f(t)$, and $L^{-1}\{G(s)\} = g(t)$ then $L^{-1}\{F(s)G(s)\} = \int_0^t f(u)g(t-u)du = F \times G$ [$F \times G$ is called the convolution or falting of F and G]**Proof :**Let $\phi(t) = \int_0^t f(u)g(t-u)du$

$$\begin{aligned} L\{\phi(t)\} &= \int_0^\infty e^{-st} \left\{ \int_0^t f(u)g(t-u)du \right\} dt \\ &= \int_0^\infty \int_0^t e^{-st} f(u)g(t-u) du dt \end{aligned}$$

The domain of integration for this double integral is the entire area lying between the lines $u = 0$ and $u = t$ as shown in Fig. 2.27.1.**Fig. 2.27.1.**

On changing the order of integration, we get

$$\begin{aligned} L\{\phi(t)\} &= \int_0^\infty \int_u^\infty e^{-st} f(u)g(t-u) dt du \\ &= \int_0^\infty e^{-su} f(u) \left\{ \int_u^\infty e^{-s(t-u)} g(t-u) dt \right\} du \\ &= \int_0^\infty e^{-su} f(u) \left\{ \int_0^\infty e^{-sv} g(v) dv \right\} du \quad [\text{on putting, } t-u=v] \\ &= \int_0^\infty e^{-su} f(u) G(s) du = \left[\int_0^\infty e^{-su} f(u) du \right] G(s) \\ &= F(s) G(s) \end{aligned}$$

Que 2.28. Apply convolution theorem to evaluate

i. $L^{-1} \frac{s}{(s^2 + a^2)^2}$

ii. $L^{-1} \frac{s^2}{(s^2 + a^2)(s^2 + b^2)}$

Answer

i. Since $f(t) = L^{-1} \left(\frac{s}{s^2 + a^2} \right) = \cos at$

and $g(t) = L^{-1} \left(\frac{1}{s^2 + a^2} \right) = \frac{1}{a} \sin at$

 $\therefore$ By convolution theorem, we get

$$\begin{aligned} L^{-1} \left[\frac{s}{s^2 + a^2} \cdot \frac{1}{s^2 + a^2} \right] &= \int_0^t \cos au \frac{\sin a(t-u)}{a} du \\ &= \frac{1}{2a} \int_0^t [\sin at - \sin(2au - at)] du \\ &= \frac{1}{2a} \left[u \sin at + \frac{1}{2a} \cos(2au - at) \right]_0^t = \frac{1}{2a} t \sin at \end{aligned}$$

Hence $L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\} = \frac{1}{2a} t \sin at$

ii. Since $f(t) = L^{-1} \left(\frac{s}{s^2 + a^2} \right) = \cos at$

and $g(t) = L^{-1} \left(\frac{s}{s^2 + b^2} \right) = \cos bt$

 $\therefore$ By convolution theorem, we get

$$\begin{aligned} L^{-1} \left(\frac{s}{s^2 + a^2} \cdot \frac{s}{s^2 + b^2} \right) &= \int_0^t \cos au \cos b(t-u) du \\ &= \frac{1}{2} \int_0^t [\cos((a-b)u + bt) + \cos((a+b)u - bt)] du \\ &= \frac{1}{2} \left[\frac{\sin((a-b)u + bt)}{a-b} + \frac{\sin((a+b)u - bt)}{a+b} \right]_0^t \\ &= \frac{1}{2} \left\{ \frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b} \right\} \\ &= \frac{a \sin at - b \sin bt}{a^2 - b^2} \end{aligned}$$

Que 2.29. Evaluate $L^{-1} \left\{ \frac{s}{(s^2 + 1)(s^2 + 4)} \right\}$

Answer

Let, $\frac{s}{(s^2 + 1)(s^2 + 4)} = \frac{1}{s^2 + 1} \cdot \frac{s}{s^2 + 4} = F(s) G(s)$

Then $f(t) = L^{-1}[F(s)] = L^{-1} \left[\frac{1}{s^2 + 1} \right] = \sin t$

$g(t) = L^{-1}[G(s)] = L^{-1} \left[\frac{s}{s^2 + 4} \right] = \cos 2t$

$\therefore$ By convolution theorem,

$L^{-1}[F(s)G(s)] = \sin t \cos 2t$

$= \int_0^t \sin u \cos 2(t-u) du$

$= \frac{1}{2} \int_0^t [\sin(2t-u) - \sin(2t-3u)] du$

$= \frac{1}{2} \left[\frac{-\cos(2t-u)}{(-1)} + \frac{\cos(2t-3u)}{(-3)} \right]_0^t$

$= \frac{1}{2} \left[\cos t - \frac{1}{3} \cos t - \cos 2t + \frac{1}{3} \cos 2t \right]$

$= \frac{1}{3} \cos t - \frac{1}{3} \cos 2t = \frac{1}{3} (\cos t - \cos 2t)$

PART-6

Application of Laplace Transform to Solve Ordinary Differential Equations and Simultaneous Differential Equations.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.30. Solve by the method of transforms, the equation

$y''' + 2y'' - y' - 2y = 0$ given $y(0) = y'(0) = 0$ and $y''(0) = 6$.

Answer

Taking the Laplace transform of both sides, we get

$[s^3 \bar{y} - s^2 y(0) - sy'(0) - y''(0)] + 2[s^2 \bar{y} - sy(0) - y'(0)] - [s\bar{y} - y(0)] - 2\bar{y} = 0$

Using the given conditions, it reduces to

$(s^3 + 2s^2 - s - 2)\bar{y} = 6$

$\therefore \bar{y} = \frac{6}{(s-1)(s+1)(s+2)} = \frac{1}{(s-1)} + \frac{-3}{(s+1)} + \frac{2}{(s+2)}$

On inversion, we get $y = L^{-1} \left[\frac{1}{(s-1)} \right] - 3L^{-1} \left[\frac{1}{(s+1)} \right] + 2L^{-1} \left[\frac{1}{(s+2)} \right]$

or $y = e^t - 3e^{-t} + 2e^{-2t}$

Que 2.31. Use transform method to solve

$\frac{d^3 x}{dt^3} - 2 \frac{dx}{dt} + x = e^t$ with $x = 2, \frac{dx}{dt} = -1$ at $t = 0$.

Answer

Taking the Laplace transforms of both sides, we get

$[s^3 \bar{x} - sx(0) - x'(0)] - 2[s\bar{x} - x(0)] + \bar{x} = \frac{1}{s-1}$

Using the given conditions, it reduces to

$(s^3 - 2s + 1)\bar{x} = \frac{1}{s-1} + 2s - 5 = \frac{2s^3 - 7s + 6}{s-1}$

$\therefore \bar{x} = \frac{2s^3 - 7s + 6}{(s-1)^3} = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$

On inversion, we obtain $x = 2L^{-1} \left[\frac{1}{s-1} \right] - 3L^{-1} \left[\frac{1}{(s-1)^2} \right] + L^{-1} \left[\frac{1}{(s-1)^3} \right]$

$= 2e^t - \frac{3e^t \cdot t}{1!} + \frac{e^t \cdot t^2}{2!} = 2e^t - 3te^t + \frac{1}{2}t^2 e^t$

Que 2.32. Solve $ty'' + 2y' + ty = \cos t$ given that $y(0) = 1$.

Answer

Taking Laplace transform of both sides of the equation

$$L[t f(t)] = -\frac{d}{ds}[L\{f(t)\}], \text{ we get}$$

$$-\frac{d}{ds}[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] - \frac{d}{ds}(\bar{y}) = \frac{s}{s^2 + 1}$$

$$\text{or } -\left(s^2 \frac{d\bar{y}}{ds} + 2s\bar{y}\right) + y(0) + 0 + 2s\bar{y} - 2y(0) - \frac{d}{ds}(\bar{y}) = \frac{s}{s^2 + 1}$$

$$\text{or } (s^2 + 1) \frac{d\bar{y}}{ds} + 1 = -\frac{s}{s^2 + 1} \text{ or } \frac{d\bar{y}}{ds} = -\frac{1}{s^2 + 1} - \frac{s}{(s^2 + 1)^2}$$

On inversion and noting that $L^{-1}\left\{\frac{1}{s^2 + 1}\right\} = -t f(t)$, we get

$$-ty = -\sin t - \frac{1}{2}t \sin t$$

or

$$y = \frac{1}{2}\left(1 + \frac{2}{t}\right) \sin t$$

Que 2.33. Solve the simultaneous equations $\frac{dx}{dt} + 5x - 2y = t$,

$\frac{dy}{dt} + 2x + y = 0$ being given $x = y = 0$ when $t = 0$.

Answer

Taking the Laplace transforms of the given equations, we get

$$[s\bar{x} - x(0)] + 5\bar{x} - 2\bar{y} = 1/s^2$$

$$(s + 5)\bar{x} - 2\bar{y} = 1/s^2 \quad \dots(2.33.1)$$

$$\text{and } s\bar{y} - y(0) + 2\bar{x} + \bar{y} = 0$$

$$2\bar{x} + (s + 1)\bar{y} = 0 \quad \dots(2.33.2)$$

Solving eq. (2.33.1) and (2.33.2) for $\bar{x}$, we get

$$\bar{x} = \begin{vmatrix} 1/s^2 & -2 \\ 0 & s+1 \end{vmatrix} + \begin{vmatrix} s+5 & -2 \\ 2 & s+1 \end{vmatrix}$$

$$= \frac{s+1}{s^2(s+3)^2} = \frac{1}{27s} + \frac{1}{9s^2} - \frac{1}{27(s+3)} - \frac{2}{9(s+3)^2}$$

Substituting the value of $\bar{x}$ in eq. (2.33.2), we get

$$\bar{y} = -\frac{2}{s^2(s+3)^2}$$

$$= \frac{4}{27s} - \frac{2}{9s^2} - \frac{4}{27(s+3)} - \frac{2}{9(s+3)^2}$$

On inversion, we get

$$x = \frac{1}{27} + \frac{t}{9} - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

$$y = \frac{4}{27} - \frac{2t}{9} - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

Que 2.34. The small oscillations of a certain system with two degrees of freedom are given by the equations:

$$D^2x + 3x - 2y = 0, D^2y + 3y - 2x = 0$$

where $D = d/dt$. If $x = 0, y = 0, \dot{x} = 3, \dot{y} = 2$ when $t = 0$, find x and y when $t = 1/2$.

Answer

Taking the Laplace transform of both the equations, we get

$$[s^2\bar{x} - sx(0) - x'(0)] + 3\bar{x} - 2\bar{y} = 0$$

$$(s^2 + 3)\bar{x} - 2\bar{y} = 3 \quad \dots(2.34.1)$$

$$\text{and } [s^2\bar{y} - sy(0) - y'(0)] + 3\bar{y} - 2\bar{x} = 0$$

$$(s^2 - 3)\bar{x} + (s^2 + 5)\bar{y} = 5 \quad \dots(2.34.2)$$

Solving eq. (2.34.1) and (2.34.2) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \begin{vmatrix} 3 & -2 \\ 5 & s^2 + 5 \end{vmatrix} + \begin{vmatrix} s^2 + 3 & -2 \\ s^2 - 3 & s^2 + 5 \end{vmatrix}$$

$$= \frac{3s^2 + 25}{(s^2 + 1)(s^2 + 9)}$$

$$= \frac{11}{4} \left(\frac{1}{s^2+1} \right) + \frac{1}{4} \left(\frac{1}{s^2+9} \right)$$

and

$$\bar{y} = \left| \begin{array}{cc} s^2+3 & 3 \\ s^2-3 & 5 \end{array} \right| \div \left| \begin{array}{cc} s^2+3 & -2 \\ s^2-3 & s^2+5 \end{array} \right| = \frac{2s^2+24}{(s^2+1)(s^2+9)}$$

$$= \frac{11}{4} \left(\frac{1}{s^2+1} \right) + \frac{3}{4} \left(\frac{1}{s^2+9} \right)$$

On inversion, we get $x = \frac{11}{4} \sin t + \frac{1}{12} \sin 3t$

$$y = \frac{11}{4} \sin t - \frac{1}{4} \sin 3t$$



Sequence and Series

CONTENTS

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PART-1**Definition of Sequence and Series with Examples,
Convergence of Series.****CONCEPT OUTLINE**

Sequence : An ordered set of real number $a_1, a_2, a_3, \dots, a_n$ is called a sequence and is denoted by (a_n) . If the number of terms is unlimited, then the sequence is said to be an infinite sequence and a_n is its general term.

Series : If $u_1, u_2, u_3, \dots, u_n, \dots$ be an infinite sequence of real numbers, then

$$u_1 + u_2 + u_3 + \dots + u_n + \dots \infty$$

is called an infinite series. An infinite series is denoted by Σu_n and the sum of its first n terms is denoted by s_n .

Convergence, Divergence and Oscillation of a Sequence :

If $\lim_{n \rightarrow \infty} (a_n) = l$ is finite and unique, the sequence is said to be convergent.

If $\lim_{n \rightarrow \infty} (a_n)$ is infinite ($\pm \infty$), the sequence is said to be divergent.

If $\lim_{n \rightarrow \infty} (a_n)$ is not unique, the sequence is said to be oscillatory.

Convergence, Divergence and Oscillation of a Series : Consider the infinite series $\Sigma u_n = u_1 + u_2 + u_3 + \dots + u_n + \dots \infty$

and let the sum of the first n terms be $s_n = u_1 + u_2 + u_3 + \dots + u_n$

Clearly, s_n is a function of n and as n increases indefinitely three possibilities arise:

- If s_n tends to a finite limit as $s_n \rightarrow \infty$, the series Σu_n is said to be convergent.
- If s_n tends to $\pm \infty$ as $n \rightarrow \infty$, the series Σu_n is said to be divergent.
- If s_n does not tend to a unique limit as $n \rightarrow \infty$, then the series Σu_n is said to be oscillatory or non-convergent.

Questions-Answers**Long Answer Type and Medium Answer Type Questions**

Que 3.1. Examine the following sequence for convergence :

- $a_n = \frac{n^2 - 2n}{3n^2 + n}$
- $a_n = 2^n$
- $a_n = 3 + (-1)^n$

Answer

i. $\lim_{n \rightarrow \infty} \left(\frac{n^2 - 2n}{3n^2 + n} \right) = \lim_{n \rightarrow \infty} \frac{1 - 2/n}{3 + 1/n} = 1/3$ which is finite and unique. Hence the sequence (a_n) is convergent.

ii. $\lim_{n \rightarrow \infty} (2^n) = \infty$. Hence the sequence (a_n) is divergent.

iii. $\lim_{n \rightarrow \infty} [3 + (-1)^n] = 3 + 1 = 4$, when n is even

$$= 3 - 1 = 2, \text{ when } n \text{ is odd}$$

i.e., this sequence doesn't have a unique limit. Hence it oscillates.

Que 3.2. Examine the following series for convergence :

i. $1 + 2 + 3 + \dots + n + \dots \infty$

ii. $5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \dots \infty$

Answer

i. Here, $s_n = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$

$$\therefore \lim_{n \rightarrow \infty} s_n = \frac{1}{2} \lim_{n \rightarrow \infty} n(n+1) \rightarrow \infty. \text{ Hence this series is divergent.}$$

ii. Here, $s_n = 5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \dots n \text{ terms}$
 $= 0, 5 \text{ or } 1$

Clearly in this case, s_n does not tend to a unique limit. Hence the series is oscillatory.

Que 3.3. Test the following series for convergence :

i. $\frac{1}{1 \cdot 2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{5}{3 \cdot 4 \cdot 5} + \dots \infty$

ii. $1 + \frac{1}{2^3} + \frac{2^3}{3^3} + \frac{3^3}{4^3} + \dots \infty$

Answer

i. We have $u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{1}{n^3} \frac{2-1/n}{(1+1/n)(1+2/n)}$

Taking $v_n = 1/n^2$, we have

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{2-1/n}{(1+1/n)(1+2/n)} = \frac{2-0}{(1+0)(1+0)}$$

$= 2$, which is finite and non zero.

Hence, both $\sum u_n$ and $\sum v_n$ converge or diverge together but $\sum v_n = \sum 1/n^2$ is known to be convergent. Hence $\sum u_n$ is also convergent.

ii. Here $u_n = \frac{n^n}{(n+1)(n+1)^n} = \frac{1}{n+1} \left(\frac{n}{n+1} \right)^n$, ignoring the first term.

Taking $v_n = 1/n$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{u_n}{v_n} \right) &= \lim_{n \rightarrow \infty} \frac{n}{n+1} \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{1+1/n} \right) \lim_{n \rightarrow \infty} \frac{1}{(1+1/n)^n} = 1 \neq 0 \end{aligned}$$

Now since $\sum v_n$ is divergent, therefore $\sum u_n$ is also divergent.

PART-2

Tests for Convergence of Series, Ratio Test, D'Alembert's Test, Raabe's Test, Comparison Test.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.4. Discuss in detail about D'Alembert's test or ratio test. Also give its limitations.

Answer

A. D'Alembert's Test or Ratio Test:
In a positive term series $\sum u_n$, if

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lambda, \text{ then the series converges for } \lambda < 1 \text{ and diverges for } \lambda > 1.$$

Case I: When, $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lambda < 1$

By definition of a limit, we can find a positive number $r (< 1)$ such that

$$\frac{u_{n+1}}{u_n} < r \text{ for all } n > m.$$

Leaving out the first m terms, let the series be $u_1 + u_2 + u_3 + \dots$

So that $\frac{u_2}{u_1} < r$, $\frac{u_3}{u_2} < r$, $\frac{u_4}{u_3} < r$, ... and so on. Then $u_1 + u_2 + u_3 + \dots \infty$

$$\begin{aligned} &= u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \frac{u_2}{u_1} + \frac{u_4}{u_3} \frac{u_3}{u_2} \frac{u_2}{u_1} + \dots \infty \right) \\ &< u_1 (1 + r + r^2 + r^3 + \dots \infty) \end{aligned}$$

$$= \frac{u_1}{1-r}, \text{ which is finite quantity. Hence } \sum u_n \text{ is convergent. } [\because r < 1]$$

Case II: When, $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lambda > 1$

By definition of limit, we can find m , such that $\frac{u_{n+1}}{u_n} \geq 1$ for all $n \geq m$.

Leaving out the first m terms, let the series be

$$u_1 + u_2 + u_3 + \dots \text{ so that } \frac{u_2}{u_1} \geq 1, \frac{u_3}{u_2} \geq 1, \frac{u_4}{u_3} \geq 1, \dots \text{ and so on.}$$

$$\begin{aligned} \therefore u_1 + u_2 + u_3 + u_4 + \dots + u_n &= u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \frac{u_2}{u_1} + \dots \right) \\ &\geq u_1 (1 + 1 + 1 + \dots \text{ to } n \text{ terms}) = nu_1 \end{aligned}$$

$\therefore \lim_{n \rightarrow \infty} (u_1 + u_2 + \dots + u_n) \geq \lim_{n \rightarrow \infty} (nu_1)$, which tends to infinity. Hence $\sum u_n$ is divergent

B. Limitations of D'Alembert's Test:

1. Ratio test fails when $\lambda = 1$.
2. This test makes no reference to the magnitude of u_{n+1}/u_n but concerns only with the limit of this ratio.

Que 3.5. State D'Alembert's test. Test the series

$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \dots + \frac{x^n}{n^2 + 1} + \dots$$

AKTU 2018-19, Marks 10

Answer

A. D'Alembert's Test: Refer Q. 3.4, Page 3-4F, Unit-3.

B. Numerical:

Leaving the first term of given series $\sum u_n$, we have

$$u_n = \frac{x^n}{n^2 + 1} \Rightarrow u_{n+1} = \frac{x^{n+1}}{n^2 + 2n + 2}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{x^n}{n^2 + 1} \times \frac{n^2 + 2n + 2}{x^{n+1}} = \frac{n^2 + 2n + 2}{n^2 + 1} \times \frac{1}{x}$$

$$= \frac{1 + (2/n) + (2/n^2)}{1 + (1/n^2)} \times \frac{1}{x},$$

Dividing each term by $n^2 \Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$

Now from Ratio Test, we conclude that

If $(1/x) > 1$ or $x < 1$, the given series $\sum u_n$ is convergent,

If $(1/x) < 1$ or $x > 1$, the given series $\sum u_n$ is divergent,

and if $1/x = 1$ or $x = 1$, this test fails and the given series becomes

$$1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \dots + \frac{1}{n^2 + 1} + \dots$$

Again leaving first term, its n^{th} term $= \frac{1}{n^2 + 1} = u_n$, say.

Then we choose $\sum v_n$ such that $v_n = \frac{1}{n^2}$

$$\text{Now, } \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1/(n^2 + 1)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{1 + (1/n^2)}$$

$$= 1, \text{ which is finite and non zero.}$$

Hence $\sum u_n$ and $\sum v_n$ both converge and diverge together. But series $\sum v_n = \sum (1/n^2)$ is convergent series (the series $\sum (1/n^p)$ is convergent if $p > 1$ and here $p = 2$). Hence this series is convergent. Thus the given series is convergent if $x \leq 1$ and divergent if $x > 1$.

Que 3.6. Test the series $\frac{x}{1.2} + \frac{x^3}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \dots$

AKTU 2021-22, Marks 10

Answer

Given series is

$$\frac{x}{1.2} + \frac{x^3}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \dots$$

Here, $u_n = \frac{x^n}{2n(2n-1)}$

If $L < 1$, then the series converges

If $L > 1$, then the series diverges

If $L = 1$, then the ratio test is inconclusive

where, $L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right|$

Applying ratio test, we get

$$L = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(2n+1)(2n+2)}}{\frac{x^n}{2n(2n-1)}} \right|$$

On simplifying this, we get

$$L = \lim_{n \rightarrow \infty} \left| \frac{x(2n-1)}{2n+2} \right|$$

$$L = \lim_{n \rightarrow \infty} \left| \frac{x}{1 + \frac{1}{n}} \right|$$

$$L = |x|$$

For convergence, $L < 1$

$$\text{i.e., } |x| < 1$$

$$\text{i.e., } -1 < x < 1$$

Thus, the series converges for $x \leq 1$ and diverges for $x > 1$.

Que 3.7. Test for convergence of the series $\frac{a+x}{1!} + \frac{(a+2x)^2}{2!}$

$$+ \frac{(a+3x)^3}{3!} + \dots$$

AKTU 2021-22, Marks 10

Answer

Let the series by $\sum u_n$.

$$\therefore u_n = \frac{(a+nx)^n}{n!}$$

$$u_{n+1} = \frac{[a + (n+1)x]^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{(a+nx)^n (n+1)}{[a + (n+1)x]^{n+1}} = \frac{(a+nx)^n}{[a + (n+1)x]^n} \cdot \frac{n+1}{a + (n+1)x}$$

$$= \frac{\left(1 + \frac{a}{nx}\right)^n}{\left(1 + \frac{a+x}{nx}\right)^n} \cdot \frac{\left(1 + \frac{1}{n}\right)}{\left[1 + \frac{a+x}{nx}\right]}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{\left[\left(1 + \frac{a}{nx}\right)^{\frac{nx}{a}}\right]^{\frac{a}{x}} \cdot \left(1 + \frac{1}{n}\right)}{\left[\left(1 + \frac{a+x}{nx}\right)^{\frac{nx}{a+x}}\right]^{\frac{a+x}{x}} \cdot \left[1 + \frac{a+x}{nx}\right]}$$

$$= \frac{e^{\frac{a}{x}}}{e^{1 + \frac{a}{x}}} \cdot \frac{1}{x(1)} = \frac{1}{xe}$$

∴ By D' Alembert's ratio test, the series converges if $\frac{1}{xe} > 1$, i.e.,

$0 < x < \frac{1}{e}$ and diverges if $x > \frac{1}{e}$.

Que 3.8. Discuss the convergence of the series.

$$1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \dots \infty$$

Answer

Given series is,

$$\sum u_n = \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

$$\text{Here, } \frac{u_n}{u_{n+1}} = \left(\frac{n!}{n^n}\right) \left(\frac{(n+1)^{n+1}}{(n+1)!}\right) = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n$$

∴ $\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$, which is > 1 . Hence the given series is convergent.

Que 3.9. Examine the convergence of the series :

$$\frac{x}{1+x} + \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} + \dots \infty$$

Answer

$$\text{Here, } u_n = \frac{x^n}{1+x^n} \text{ and } u_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \left(\frac{x^n}{x^{n+1}} \cdot \frac{1+x^{n+1}}{1+x^n} \right) = \lim_{n \rightarrow \infty} \left(\frac{1+x^{n+1}}{x+x^{n+1}} \right)$$

$$= \frac{1}{x}, \text{ if } x < 1 \quad [\because x^{n+1} \rightarrow 0 \text{ and } n \rightarrow \infty]$$

$$\text{Also, } \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \left(\frac{1 + 1/x^{n+1}}{1 + x/x^{n+1}} \right) = 1, \text{ if } x > 1.$$

By ratio test, $\sum u_n$ converges for $x < 1$ and fails for $x \geq 1$.

When $x = 1$, $\sum u_n = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \infty$, which is divergent.

Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

Que 3.10. Explain Raabe's test in brief.

Answer

In the positive term series $\sum u_n$, if $\lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = k$, then the series converges for $k > 1$ and diverges for $k < 1$, but the test fails for $k = 1$. When $k > 1$, choose a number p such that $k > p > 1$, and compare $\sum u_n$ with the series $\sum \frac{1}{n^p}$ which is convergent since $p > 1$.

∴ $\sum u_n$ will converge, if from and after some term,

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} \text{ or } \left(1 + \frac{1}{n}\right)^p$$

$$\text{or if, } \frac{u_n}{u_{n+1}} > 1 + \frac{p}{n} + \frac{p(p-1)}{2n^2} + \dots$$

$$\text{or if, } n \left(\frac{u_n}{u_{n+1}} - 1 \right) > p + \frac{p(p-1)}{2n} + \dots$$

$$\text{or if, } \lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) > \lim_{n \rightarrow \infty} \left[p + \frac{p(p-1)}{2n} + \dots \right]$$

i.e., if $k > p$, which is true. Hence, Σu_n is convergent.

The other case when $k < 1$ can be proved similarly.

Que 3.11. Test for convergence of the following the series :

i. $\sum \frac{4 \cdot 7 \dots (3n+1)}{1 \cdot 2 \dots n} x^{3n}$ ii. $\sum \frac{(n!)^2}{(2n)!} x^{2n}$

Answer

i. Here, $\frac{u_n}{u_{n+1}} = \frac{4 \cdot 7 \dots (3n+1)}{1 \cdot 2 \dots n} x^{3n} \div \frac{4 \cdot 7 \dots (3n+4)}{1 \cdot 2 \dots (n+1)} x^{3(n+1)} = \left(\frac{n+1}{3n+4} \right) \left(\frac{1}{x} \right)$

$$= \left[\frac{1+1/n}{3+4/n} \right] \frac{1}{x}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1}{3x}$$

Thus by ratio test, the series converges for $\frac{1}{3x} > 1$, i.e. for $x < \frac{1}{3}$ and diverges for $x > \frac{1}{3}$. But it fails for $x = \frac{1}{3}$.

$\therefore$ Let us try the Raabe's test

Now, $\frac{u_n}{u_{n+1}} = \left(1 + \frac{1}{n} \right) \left(1 + \frac{4}{3n} \right)^{-1}$ [Expand by binomial theorem]

$$= \left(1 + \frac{1}{n} \right) \left(1 - \frac{4}{3n} + \frac{16}{9n^2} - \dots \right) = 1 - \frac{1}{3n} + \frac{4}{9n^2} + \dots$$

$$\therefore n \left(\frac{u_n}{u_{n+1}} - 1 \right) = -\frac{1}{3} + \frac{4}{9n} + \dots$$

$$\therefore \lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = -\frac{1}{3} \text{ which is } < 1$$

Thus by Raabe's test, the series diverges.

Hence the given series converges for $x < (1/3)$ and diverges for $x \geq (1/3)$.

ii. Here, $\frac{u_n}{u_{n+1}} = \left(\frac{n!}{(n+1)!} \right)^2 \frac{[2(n+1)!]}{(2n)!} \frac{x^{2n}}{x^{2(n+1)}}$

$$= \frac{(2n+1)(2n+2)}{(n+1)^2} \frac{1}{x^2} = \frac{2(2n+1)}{n+1} \frac{1}{x^2}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{2(2+1/n)}{1+1/n} \frac{1}{x^2} = \frac{4}{x^2}$$

Thus by ratio test, the series converges for $x^2 < 4$ and diverges for $x^2 > 4$. But fails for $x^2 = 4$.

When $x^2 = 4$, $n \left(\frac{u_n}{u_{n+1}} - 1 \right) = n \left(\frac{2n+1}{2n+2} - 1 \right) = -\frac{n}{2n+2}$

$$\therefore \lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = -\frac{1}{2} < 1$$

Thus by Raabe's test, the series diverges.

Hence the given series converges for $x^2 < 4$ and diverges for $x^2 \geq 4$.

Que 3.12. Explain comparison test in brief.

Answer

Comparison test :

1. If two positive term series Σu_n and Σv_n be such that

i. Σv_n converges

ii. $u_n \leq v_n$ for all values of n , then Σu_n also converges.

Proof : Since Σv_n is convergent,

$$\lim_{n \rightarrow \infty} (v_1 + v_2 + v_3 + \dots + v_n) = \text{a finite quantity } k \text{ (say)}$$

Also since $u_1 \leq v_1, u_2 \leq v_2, \dots, u_n \leq v_n$

$$\therefore \text{Adding, } u_1 + u_2 + \dots + u_n \leq v_1 + v_2 + \dots + v_n$$

$$\therefore \lim_{n \rightarrow \infty} (u_1 + u_2 + \dots + u_n) \leq \lim_{n \rightarrow \infty} (v_1 + v_2 + \dots + v_n) = k.$$

Hence the series Σu_n also converges.

2. If two positive term series Σu_n and Σv_n be such that

i. Σv_n diverges,

ii. $u_n \geq v_n$ for all values of n , then Σu_n is also diverges.

3. Limit form :

If two positive term series Σu_n and Σv_n be such that

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \text{finite quantity } (\neq 0)$, then Σu_n and Σv_n converge or diverge together.

Proof : Since $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = l$, a finite number ($\neq 0$)

By definition of a limit, there exists a positive number ε , however small, such that

$$\left| \frac{u_n}{v_n} - l \right| < \varepsilon \quad \text{for } n \geq m$$

$$\text{or} \quad -\varepsilon < \frac{u_n}{v_n} - l < \varepsilon \quad \text{for } n \geq m$$

$$\text{or} \quad l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \quad \text{for } n \geq m$$

Omitting the first n terms of both the series, we have

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \quad \text{for all } n \quad \dots(3.12.1)$$

Case 1 : When $\sum v_n$ is convergent, then

$$\lim_{n \rightarrow \infty} (v_1 + v_2 + \dots + v_n) = k, \text{ a finite number} \quad (3.12.2)$$

Also from (3.12.1),

$$\frac{u_n}{v_n} < l + \varepsilon, \text{ i.e., } u_n < (l + \varepsilon) v_n \text{ for all } n.$$

$$\therefore \lim_{n \rightarrow \infty} (u_1 + u_2 + \dots + u_n) < (l + \varepsilon) \lim_{n \rightarrow \infty} (v_1 + v_2 + \dots + v_n) = (l + \varepsilon) k$$

Hence $\sum u_n$ is also convergent.

Case 2 : When $\sum v_n$ is divergent, then

$$\lim_{n \rightarrow \infty} (v_1 + v_2 + \dots + v_n) \rightarrow \infty$$

$$\text{Also (3.12.1) } l - \varepsilon < \frac{u_n}{v_n} \text{ or } u_n > (l - \varepsilon) v_n \text{ for all } n$$

$$\therefore \lim_{n \rightarrow \infty} (u_1 + u_2 + \dots + u_n) > (l - \varepsilon) \lim_{n \rightarrow \infty} (v_1 + v_2 + \dots + v_n) \rightarrow \infty$$

Hence $\sum u_n$ is also divergent.

Que 3.13. | Test the convergence of the series :

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{(n+1)}}$$

Answer

We have

$$\begin{aligned} u_n &= \frac{\sqrt{(n+1)} - \sqrt{n}}{[\sqrt{(n+1)} + \sqrt{n}][\sqrt{(n+1)} - \sqrt{n}]} \\ &= \frac{\sqrt{(n+1)} - \sqrt{n}}{(\sqrt{(n+1)} + \sqrt{n})} \\ &= \frac{1}{\sqrt{n}} \left(\frac{1}{2} - \frac{1}{8n} + \dots \right) \\ &= \frac{1}{\sqrt{n}} \left(\frac{1}{2} - \frac{1}{8n} + \dots \right) \end{aligned}$$

Taking $v_n = 1/\sqrt{n}$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{u_n}{v_n} &= \lim_{n \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{8n} + \dots \right) \\ &= \frac{1}{2}, \text{ which is finite and non zero.} \end{aligned}$$

$\therefore$ Both $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n = \sum 1/\sqrt{n}$ is known to be divergent. Hence $\sum u_n$ is also divergent.

PART-3

Fourier Series.

CONCERT OUTLINE

Fourier Series in the Interval $C < x < C + 2\pi$: The Fourier series for the function $f(x)$ in the interval $C < x < C + 2\pi$ is given as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where a_0 , a_n and b_n are called Fourier coefficients, and given as

$$a_0 = \frac{1}{\pi} \int_C^{C+2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_C^{C+2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_C^{C+2\pi} f(x) \sin nx dx$$

Fourier Series when Interval is Changed : Fourier series in the interval $C < x < C + 2L$ is given as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Where, $a_0 = \frac{1}{L} \int_C^{C+2L} f(x) dx$

$$a_n = \frac{1}{L} \int_C^{C+2L} f(x) \cos \frac{n\pi x}{L} dx$$

and $b_n = \frac{1}{L} \int_C^{C+2L} f(x) \sin \frac{n\pi x}{L} dx$

Note:

i. If $C = -L$, then interval is $-L < x < L$ and

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \left(\frac{n\pi x}{L} \right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

ii. If $f(x)$ is an odd function then,

$$a_n = a_0 = 0.$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

iii. If $f(x)$ is an even function then,

$$b_n = 0 \text{ and } a_0 = \frac{2}{L} \int_0^L f(x) dx,$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \left(\frac{n\pi x}{L} \right) dx,$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.14. Find the Fourier series expansion of the function

$$f(x) = \begin{cases} -1, & \text{for } -\pi < x < -\pi/2 \\ 0, & \text{for } -\pi/2 < x < \pi/2 \\ 1, & \text{for } \pi/2 < x < \pi \end{cases}$$

Hence deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

AKTU 2011-12, Marks 10

Answer

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^{-\pi/2} (-1) dx + \int_{\pi/2}^{\pi} (1) dx \right]$$

$$= \frac{1}{\pi} \left[-\left(-\frac{\pi}{2} + \pi\right) + \left(\pi - \frac{\pi}{2}\right) \right]$$

$$a_0 = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{-\pi/2} -\cos nx dx + \int_{\pi/2}^{\pi} \cos nx dx \right]$$

$$= \frac{1}{\pi} \left[-\left\{ \frac{\sin nx}{n} \right\}_{-\pi}^{-\pi/2} - \left\{ \frac{\sin nx}{n} \right\}_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{\sin n\pi/2}{n} - \frac{\sin n\pi}{n} - \frac{\sin n\pi}{n} + \frac{\sin n\pi/2}{n} \right]$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{-\pi/2} -\sin nx dx + \int_{\pi/2}^{\pi} \sin nx dx \right]$$

$$= \frac{1}{\pi} \left[\left\{ \frac{\cos nx}{n} \right\}_{-\pi}^{-\pi/2} - \left\{ \frac{\cos nx}{n} \right\}_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{\cos n\pi/2}{n} - \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} + \frac{\cos n\pi/2}{n} \right]$$

$$b_n = \frac{2}{\pi n} \left[\cos \frac{n\pi}{2} - \cos n\pi \right]$$

Hence required series is,

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(\cos \frac{n\pi}{2} - \cos n\pi \right) \sin nx$$

Putting $x = \pi/2$ in the above series,

$$[f(x)]_{x=\pi/2} = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(\cos \frac{n\pi}{2} - \cos n\pi \right) \sin \frac{n\pi}{2}$$

$$\frac{0+1}{2} = \frac{2}{\pi} \sum \frac{1}{n} \left(\cos \frac{n\pi}{2} - \cos n\pi \right) \sin \frac{n\pi}{2}$$

Putting $n = 1, 2, 3, 4, \dots$

$$\begin{aligned} \frac{\pi}{4} &= \frac{1}{1} \left(\cos \frac{\pi}{2} - \cos \pi \right) \sin \frac{\pi}{2} + 0 \\ &+ \frac{1}{3} \left(\cos \frac{3\pi}{2} - \cos 3\pi \right) \sin \frac{3\pi}{2} + 0 \\ &+ \frac{1}{5} \left(\cos \frac{5\pi}{2} - \cos 5\pi \right) \sin \frac{5\pi}{2} + 0 + \dots \end{aligned}$$

$$\frac{\pi}{4} = 1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + \dots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Que 3.15. Find the Fourier series to represent the function $f(x)$ given by

$$f(x) = \begin{cases} \pi x & ; 0 \leq x \leq 1 \\ \pi(2-x) & ; 1 \leq x \leq 2 \end{cases}$$

AKTU 2013-14, Marks 10

Answer

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

Then $a_0 = \int_0^2 f(x) dx = \int_0^1 \pi x dx + \int_1^2 \pi(2-x) dx$

$$a_0 = \pi \left[\frac{x^2}{2} \right]_0^1 + \pi \left[2x - \frac{x^2}{2} \right]_1^2 = \pi \left(\frac{1}{2} \right) + \pi \left[(4-2) - \left(2 - \frac{1}{2} \right) \right]$$

$$a_0 = \pi$$

$$a_n = \int_0^2 f(x) \cos n\pi x dx$$

$$= \int_0^1 \pi x \cos n\pi x dx + \int_1^2 \pi(2-x) \cos n\pi x dx$$

$$a_n = \left[\pi x \frac{\sin n\pi x}{n\pi} - \pi \left(-\frac{\cos n\pi x}{n^2 \pi^2} \right) \right]_0^1$$

$$+ \left[\pi(2-x) \frac{\sin n\pi x}{n\pi} - (-\pi) \left(-\frac{\cos n\pi x}{n^2 \pi^2} \right) \right]_1^2$$

$$= \left[\frac{\cos n\pi}{n^2 \pi} - \frac{1}{n^2 \pi} \right] + \left[\frac{-\cos 2n\pi}{n^2 \pi} + \frac{\cos n\pi}{n^2 \pi} \right]$$

$$= \frac{2}{n^2 \pi} [\cos n\pi - 1] = \frac{2}{n^2 \pi} [(-1)^n - 1] \quad (\because \cos 2n\pi = 1)$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{4}{n^2 \pi}, & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \int_0^2 f(x) \sin n\pi x dx$$

$$= \int_0^1 \pi x \sin n\pi x dx + \int_1^2 \pi(2-x) \sin n\pi x dx$$

$$= \left[\pi x \left(-\frac{\cos n\pi x}{n\pi} \right) - \pi \left(\frac{-\sin n\pi x}{n^2 \pi^2} \right) \right]_0^1 + \left[\pi(2-x) \left(-\frac{\cos n\pi x}{n\pi} \right) - (-\pi) \left(\frac{-\sin n\pi x}{n^2 \pi^2} \right) \right]_1^2$$

$$= \left[-\frac{\cos n\pi}{n} \right] + \left[\frac{\cos n\pi}{n} \right] = 0$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots \right)$$

Que 3.16. Express $f(x) = |x|$; $-\pi < x < \pi$ as Fourier series.

AKTU 2013-14, Marks 10

Answer

$$\text{Since } f(-x) = |-x| = |x| = f(x)$$

$$\therefore f(x) \text{ is an even function and hence } b_n = 0$$

$$\text{Let } f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\text{Where, } a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} |x| dx = \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} |x| \cos nx dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$$

$$\begin{aligned}
 &= \frac{2}{\pi} \left[x \left(\frac{\sin nx}{n} \right) - 1 \left(\frac{-\cos nx}{n^2} \right) \right]_0^{\pi} \\
 &= \frac{2}{\pi} \left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right] = \frac{2}{\pi n^2} [(-1)^n - 1] \\
 a_n &= \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases} \\
 |x| &= \frac{\pi}{2} - \frac{4}{\pi} \left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right)
 \end{aligned}$$

Que 3.17. Expand $f(x) = x \sin x$ as a Fourier series in $0 < x < 2\pi$.

AKTU 2014-15, Marks 10

Answer

$$\begin{aligned}
 f(x) &= x \sin x ; 0 < x < 2\pi \\
 a_0 &= \frac{1}{\pi} \int_0^{2\pi} x \sin x \, dx = \frac{1}{\pi} [x(-\cos x) + \sin x]_0^{2\pi} = \frac{1}{\pi} [-2\pi] \\
 a_0 &= -2 \\
 a_n &= \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos nx \, dx = \frac{1}{2\pi} \int_0^{2\pi} x (2 \cos nx \sin x) \, dx \\
 &= \frac{1}{2\pi} \int_0^{2\pi} x [\sin(1+n)x + \sin(1-n)x] \, dx \\
 &= \frac{1}{2\pi} \left[-x \frac{\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2} \right. \\
 &\quad \left. + \frac{x \cos(n-1)x}{n-1} - \frac{\sin(n-1)x}{(n-1)^2} \right]_0^{2\pi} \\
 &= \frac{1}{2\pi} \left[\frac{-2\pi}{n+1} + \frac{2\pi}{n-1} \right] = \frac{1}{n-1} - \frac{1}{n+1} \\
 a_n &= \frac{2}{n^2-1}, n \neq 1
 \end{aligned}$$

When $n = 1$, we have

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos x \, dx = \frac{1}{2\pi} \int_0^{2\pi} x \sin 2x \, dx$$

$$\begin{aligned}
 &= \frac{1}{2\pi} \left[x \left(\frac{-\cos 2x}{2} \right) + \frac{\sin 2x}{4} \right]_0^{2\pi} = \frac{1}{2\pi} \left[\frac{-2\pi}{2} \right] \\
 a_1 &= -\frac{1}{2} \\
 b_n &= \frac{1}{\pi} \int_0^{2\pi} x \sin x \sin nx \, dx \\
 &= \frac{1}{2\pi} \int_0^{2\pi} x [\cos(1-n)x - \cos(1+n)x] \, dx \\
 &= \frac{1}{2\pi} \left[x \frac{\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2} \right. \\
 &\quad \left. - \frac{x \sin(n+1)x}{n+1} - \frac{\cos(n+1)x}{(n+1)^2} \right]_0^{2\pi} \\
 &= \frac{1}{2\pi} \left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} \right]
 \end{aligned}$$

When $n = 1$, we have

$$\begin{aligned}
 b_1 &= \frac{1}{\pi} \int_0^{2\pi} x \sin x \sin x \, dx = \frac{1}{2\pi} \int_0^{2\pi} x (1 - \cos 2x) \, dx \\
 &= \frac{1}{2\pi} \left[x \left(x - \frac{\sin 2x}{2} \right) - 1 \left(\frac{x^3}{2} - \frac{\cos 2x}{4} \right) \right]_0^{2\pi} \\
 &= \frac{1}{2\pi} \left[2\pi(2\pi) - \frac{4\pi^3}{2} - \frac{1}{4} + \frac{1}{4} \right] = \frac{1}{2\pi} (2\pi^2) = \pi
 \end{aligned}$$

$$\begin{aligned}
 \therefore f(x) &= \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx \\
 &= -1 - \frac{1}{2} \cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1} \cos nx
 \end{aligned}$$

Que 3.18. Find the Fourier series to represent the function $f(x)$ given by

$$f(x) = \begin{cases} -K & \text{for } -\pi < x < 0 \\ K & \text{for } 0 < x < \pi \end{cases}$$

Hence show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$. **AKTU 2015-16, Marks 10**

Answer

$$f(x) = \begin{cases} -K & \text{for } -\pi < x < 0 \\ K & \text{for } 0 < x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 (-K) dx + \frac{1}{\pi} \int_0^{\pi} K dx$$

$$a_0 = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 -K \cos nx dx + \frac{1}{\pi} \int_0^{\pi} K \cos nx dx$$

$$a_n = -\frac{K}{\pi} \left[\frac{\sin nx}{n} \right]_{-\pi}^0 + \frac{K}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi}$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 (-K \sin nx) dx + \frac{1}{\pi} \int_0^{\pi} K \sin nx dx$$

$$= \frac{1}{\pi} \left[-K \left(\frac{-\cos nx}{n} \right) \right]_{-\pi}^0 + \frac{1}{\pi} \left[K \left(\frac{-\cos nx}{n} \right) \right]_0^{\pi}$$

$$= \frac{K}{\pi} \left[\frac{1}{n} \frac{(-1)^n}{n} - \frac{(-1)^n}{n} + \frac{1}{n} \right]$$

$$b_n = \frac{K}{\pi} \left[\frac{2}{n} \frac{(-1)^n}{n} \right]$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4K}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

$$f(x) = \frac{4K}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right]$$

Now putting $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = K = \frac{4K}{\pi} \left[1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + \dots \right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Que 3.20. Obtain Fourier series for the function

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi < x < 0 \\ 1 - \frac{2x}{\pi}, & 0 < x < \pi \end{cases}$$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$.

AKTU 2021-22, Marks 10**Answer**

Since

$$f(-x) = 1 - \frac{2x}{\pi} \text{ in } (-\pi, 0) = f(x) \text{ in } (0, \pi)$$

$$\text{and } f(-x) = 1 + \frac{2x}{\pi} \text{ in } (0, \pi) = f(x) \text{ in } (-\pi, 0)$$

 $\therefore f(x)$ is an even function in $(-\pi, \pi)$ $\therefore$

$$b_n = 0$$

 $\therefore$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

...(3.20.1)

where,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi} \right) dx$$

$$= \frac{2}{\pi} \left(x - \frac{x^2}{\pi} \right)_0^{\pi} = 0$$

and

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$\begin{aligned}
 &= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos nx \, dx \\
 &= \frac{2}{\pi} \left[\left(1 - \frac{2x}{\pi}\right) \frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right) \left(-\frac{\cos nx}{n^2}\right) \right]_0^{\pi} \\
 &= \frac{2}{\pi} \left(-\frac{2 \cos n\pi}{n^2} + \frac{2}{n^2} \right) = \frac{4}{n^2 \pi} [1 - (-1)^n]
 \end{aligned}$$

$$\therefore a_1 = 8/\pi^2, a^3 = 8/3^2 \pi^2, a_5 = 8/5^2 \pi^2, \dots$$

$$\text{and } a_2 = a_4 = a_6 = \dots = 0.$$

Thus substituting the values of a's in (3.20.1), we get

$$f(x) = \frac{8}{\pi^2} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right] \quad \dots(3.20.2)$$

as the required Fourier expansion.

$$\text{Putting } x = 0 \text{ in (3.20.2), we get } f(0) = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) \left[\because f(0) = 1 \right]$$

Que 3.21. Obtain Fourier series for the function

$$f(x) = \begin{cases} x & , -\pi < x < 0 \\ -x & , 0 < x < \pi \end{cases} \text{ and hence show that } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

AKTU 2017-18, Marks 07

Answer

Let the Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \dots(3.21.1)$$

$$\text{Now, } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left(\int_{-\pi}^0 x dx + \int_0^{\pi} -x dx \right)$$

$$= \frac{1}{\pi} \left[\left[\frac{x^2}{2} \right]_{-\pi}^0 - \left[\frac{x^2}{2} \right]_0^{\pi} \right] = \frac{1}{\pi} \left(0 - \frac{\pi^2}{2} - \frac{\pi^2}{2} \right) = -\pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 x \cos nx \, dx + \int_0^{\pi} -x \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\left[\frac{x \sin nx}{n} \right]_{-\pi}^0 - \int_{-\pi}^0 1 \frac{\sin nx}{n} dx \right]$$

$$\begin{aligned}
 &+ \left[\frac{-x \sin nx}{n} \right]_0^{\pi} - \int_0^{\pi} (-1) \frac{\sin nx}{n} dx \Big\} \\
 &= \frac{1}{\pi} \left\{ \frac{1}{n^2} (\cos nx)_0^{\pi} - \frac{1}{n^2} (\cos nx)_0^{\pi} \right\} \\
 &= \frac{1}{\pi} \left[\left\{ \frac{1 - (-1)^n}{n^2} \right\} - \left\{ \frac{(-1)^n - 1}{n^2} \right\} \right] = \frac{1}{\pi} \left[\frac{2(1 - (-1)^n)}{n^2} \right] \\
 &= \frac{2}{\pi n^2} \{1 - (-1)^n\}
 \end{aligned}$$

$$= \begin{cases} 0 & , \text{ if } n \text{ is even.} \\ \frac{4}{\pi n^2} & , \text{ if } n \text{ is odd.} \end{cases}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 x \sin nx \, dx + \int_0^{\pi} (-x) \sin nx \, dx \right]$$

$$= \frac{1}{\pi} \left\{ x \left(\frac{-\cos nx}{n} \right) \Big|_{-\pi}^0 - \int_{-\pi}^0 1 \left(\frac{-\cos nx}{n} \right) dx \right.$$

$$\left. + \left\{ (-x) \left(\frac{-\cos nx}{n} \right) \right\} \Big|_0^{\pi} - \int_0^{\pi} (-1) \left(\frac{-\cos nx}{n} \right) dx \right\}$$

$$= \frac{1}{\pi} \left[\left\{ 0 - \frac{\pi \cos n\pi}{n} \right\} + \frac{1}{n} \left[\frac{\sin nx}{n} \right]_{-\pi}^0 + \left\{ \frac{\pi(-1)^n}{n} - 0 \right\} - \frac{1}{n} \left[\frac{\sin nx}{n} \right]_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[-\frac{\pi(-1)^n}{n} + \frac{1}{n} \pi(-1)^n \right]$$

$$= 0, \text{ whatever be the value of } n.$$

Therefore, the Fourier series is

$$f(x) = \frac{-\pi}{2} + \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right] \quad \dots(3.21.2)$$

Since the function $f(x)$ is discontinuous at $x = 0$, by Dirichlet's condition

$$f(0) = \frac{1}{2} [\text{LHL} + \text{RHL}] = (1/2)[f(0-0) + f(0+0)] = 0$$

Putting $x = 0$ in eq. (3.21.2), we get

$$0 = \frac{-\pi}{2} + \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

Que 3.22. Find the Fourier series of $f(x) = x \sin x$ in $-\pi \leq x \leq \pi$

AKTU 2018-19, Marks 10

Answer

Since $x \sin x$ is an even function of $x \therefore b_n = 0$

Let $f(x) = x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Where, $a_0 = \frac{2}{\pi} \int_0^{\pi} x \sin x \, dx = \frac{2}{\pi} [x(-\cos x) - 1(-\sin x)]_0^{\pi}$

$$= \frac{2}{\pi} (-\pi \cos \pi) = 2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx = \frac{1}{\pi} \int_0^{\pi} x(2 \cos nx \sin x) \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x] \, dx$$

$$= \frac{1}{\pi} \left[x \left\{ -\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right\} \right. \\ \left. - \frac{1}{(n+1)^2} \sin(n+1)x + \frac{1}{(n-1)^2} \sin(n-1)x \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\pi \left\{ -\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} \right\} \right]$$

$$= \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}, n \neq 1$$

When n is odd, $n \neq 1$, $n-1$ and $n+1$ are even

$$\therefore a_n = \frac{1}{n-1} - \frac{1}{n+1} = \frac{2}{n^2-1}$$

When n is even, $n-1$ and $n+1$ are odd

$$\therefore a_n = \frac{-1}{n-1} + \frac{1}{n+1} = \frac{-2}{n^2-1}$$

When $n = 1$, we have

$$a_1 = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi} \left[x \left(-\frac{\cos 2x}{2} \right) - 1 \left(-\frac{\sin 2x}{4} \right) \right]_0^{\pi} = \frac{1}{\pi} \left[-\frac{\pi \cos 2\pi}{2} \right] = -\frac{1}{2}$$

$$\therefore x \sin x = 1 - \frac{1}{2} \cos x - 2 \left(\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \frac{\cos 5x}{5^2-1} + \dots \right)$$

PART-4

Half Range Fourier Sine and Cosine Series.

CONCEPT OUTLINE

Half Range Series : Half series is found when a periodic function is expanded in half range of its period i.e. to expand $f(x)$ in range $(0, L)$ having a period of $2L$.

A function $f(x)$ defined in the interval $(0, L)$ has two half range series that are called Fourier cosine and Fourier sine series.

Half Range Cosine Series : The half range cosine series is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

Where,

$$a_0 = \frac{2}{L} \int_0^L f(x) \, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx$$

Half Range Sine Series : The half range sine series is given as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Where,

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.23. Find the half range cosine series expansion of

$$f(x) = x - x^2, \quad 0 < x < 1$$

AKTU 2011-12, 2012-13; Marks 05

Answer

$$f(x) = x - x^2, \quad 0 < x < 1$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{1}$$

$$a_0 = 2 \int_0^1 f(x) dx = 2 \int_0^1 (x - x^2) dx$$

$$= 2 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = 2 \left[\frac{1}{2} - \frac{1}{3} \right] = \frac{2}{6} = \frac{1}{3}$$

$$a_n = \frac{2}{1} \int_0^1 f(x) \cos \frac{n\pi x}{1} dx = 2 \int_0^1 (x - x^2) \cos n\pi x dx$$

$$= 2 \left[(x - x^2) \left(\frac{\sin n\pi x}{n\pi} \right) - (1 - 2x) \left(\frac{-\cos n\pi x}{n^2 \pi^2} \right) + (-2) \left(\frac{-\sin n\pi x}{n^3 \pi^3} \right) \right]_0^1$$

$$= 2 \left[(-1) \frac{\cos n\pi}{n^2 \pi^2} - \frac{1}{n^2 \pi^2} \right] = 2 \left[\frac{(-1)^{n+1}}{n^2 \pi^2} - \frac{1}{n^2 \pi^2} \right]$$

$$f(x) = \frac{1}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} [(-1)^{n+1} - 1] \cos n\pi x$$

Que 3.24. Find the Fourier half range sine series for

$$f(x) = (x + 1) \text{ for } 0 < x < \pi.$$

AKTU 2013-14, Marks 05

Answer

$$f(x) = x + 1$$

$$x + 1 = \sum_{n=1}^{\infty} b_n \sin nx$$

Where,

$$b_n = \frac{2}{\pi} \int_0^{\pi} (x + 1) \sin nx dx$$

$$= \frac{2}{\pi} \left[(x + 1) \left(\frac{-\cos nx}{n} \right) \right]_0^{\pi} - \int_0^{\pi} (1) \left(\frac{-\cos nx}{n} \right) dx$$

$$= \frac{2}{\pi} \left[(\pi + 1) \left(\frac{-\cos n\pi}{n} \right) + \frac{\cos 0^\circ}{n} \right] + \left[\frac{\sin nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{(\pi + 1)(-(-1)^n) + 1}{n} + \left[\frac{\sin n\pi - \sin 0^\circ}{n^2} \right] \right]$$

$$= \frac{2}{\pi n} [1 - (1 + \pi)(-1)^n]$$

$$= \begin{cases} -\frac{2}{n}; & \text{If } n \text{ is even} \\ \frac{2}{\pi n} (2 + \pi); & \text{If } n \text{ is odd} \end{cases}$$

Here Fourier sine series is

$$f(x) = x + 1 = \frac{2(2 + \pi)}{\pi} \left[\frac{\sin x}{3} + \frac{\sin 3x}{5} + \frac{\sin 5x}{7} + \dots \right] - 2 \left[\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} + \dots \right]$$

Que 3.25. Find the half range sine expansion of

$$f(t) = \begin{cases} t & ; 0 < t < 2 \\ 4 - t & ; 2 < t < 4 \end{cases}$$

AKTU 2014-15, Marks 05

OR

Find half range sine series of $f(x) = \begin{cases} x & ; 0 < x < 2 \\ 4 - x & ; 2 < x < 4 \end{cases}$

AKTU 2018-19, Marks 10

Answer

$$b_n = \frac{1}{2} \int_0^4 f(t) \sin \frac{n\pi t}{4} dt$$

$$= \frac{1}{2} \left[\int_0^2 t \sin \left(\frac{n\pi t}{4} \right) dt + \int_2^4 (4 - t) \sin \left(\frac{n\pi t}{4} \right) dt \right]$$

$$\begin{aligned}
 &= \frac{1}{2} \left[\left\{ t \left(-\frac{4}{n\pi} \cos \frac{n\pi t}{4} \right) + \frac{16}{n^2 \pi^2} \sin \left(\frac{n\pi t}{4} \right) \right\}_0^2 \right. \\
 &\quad \left. + \left\{ (4-t) \left(-\frac{4}{n\pi} \cos \frac{n\pi t}{4} \right) - \frac{16}{n^2 \pi^2} \sin \frac{n\pi t}{4} \right\}_2^4 \right] \\
 &= \frac{1}{2} \left[-\frac{8}{n\pi} \cos \frac{n\pi}{2} + \frac{16}{n^2 \pi^2} \sin \left(\frac{n\pi}{2} \right) + \frac{8}{n\pi} \cos \left(\frac{n\pi}{2} \right) + \frac{16}{n^2 \pi^2} \sin \left(\frac{n\pi}{2} \right) \right] \\
 &= \frac{16}{n^2 \pi^2} \sin \left(\frac{n\pi}{2} \right)
 \end{aligned}$$

Hence the Fourier series is,

$$f(x) = \sum_{n=0}^{\infty} \frac{16}{n^2 \pi^2} \sin \left(\frac{n\pi}{2} \right) \sin \left(\frac{n\pi t}{4} \right)$$

Que 3.26. Obtain the Fourier expansion of $f(x) = x \sin x$ as cosine series in $(0, \pi)$ and hence show that

$$\frac{1}{1 \times 3} - \frac{1}{3 \times 5} + \frac{1}{5 \times 7} - \dots = \left(\frac{\pi - 2}{4} \right)$$

AKTU 2016-17, Marks 07

Answer

Let the Fourier series be

$$f(x) = x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Now,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin x \, dx$$

[$\because x \sin x$ is an even function]

Using $\int uv \, dx = uv_1 - u'v_2 + \dots$, we have

$$\begin{aligned}
 &= \frac{2}{\pi} [x(-\cos x) + (\sin x)]_0^{\pi} \\
 &= \frac{2}{\pi} (-\pi \cos \pi) = 2
 \end{aligned}$$

And,

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx \\
 &= \frac{1}{\pi} \int_0^{\pi} x (2 \cos nx \sin x) \, dx
 \end{aligned}$$

$$= \frac{1}{\pi} \int_0^{\pi} x [\sin(n+1)x - \sin(n-1)x] \, dx$$

$$[\because 2 \cos A \sin B = \sin(A+B) - \sin(A-B)]$$

$$= \frac{1}{\pi} \left[x \left\{ \frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right\} \right.$$

$$\left. - \left\{ \frac{-\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2} \right\} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\pi \left\{ \frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} \right\} - \{0-0\} \right]$$

$$= \frac{1}{\pi} \left[\pi \left\{ \frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} \right\} \right], n \neq 1$$

$$= -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} = (-1)^{n-1} \left[\frac{1}{n-1} - \frac{1}{n+1} \right] = \frac{2(-1)^{n-1}}{(n-1)(n+1)}$$

When $n = 1$, we have

$$a_1 = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi} \left[x \left(-\frac{\cos 2x}{2} \right) - 1 \left(-\frac{\sin 2x}{2^2} \right) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-\frac{\pi \cos 2\pi}{2} \right] = -\frac{1}{2}$$

$$\therefore x \sin x = 1 - \frac{1}{2} \cos x - 2 \left(\frac{\cos 2x}{1.3} - \frac{\cos 3x}{2.4} + \frac{\cos 4x}{3.5} - \dots \right)$$

Putting $x = \frac{\pi}{2}$, we get

$$\frac{\pi}{2} = 1 - 2 \left(\frac{-1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \dots \right)$$

$$1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \dots = \frac{\pi}{2}$$

$$\frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \dots = \frac{\pi}{2} - 1$$

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots = \frac{\pi - 2}{4}$$

Hence



4 UNIT

Complex Variable Differentiation

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PART-1

*Functions of Complex Variable, Limit,
Continuity and Differentiability.*

CONCEPT OUTLINE

Limit : The function $f(x, y)$ tends to the limit l as $x \rightarrow a$ and $y \rightarrow b$ if and only if the limit l is independent of the path followed by the point (x, y) as $x \rightarrow a$ and $y \rightarrow b$. Then

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = l$$

The function $f(x, y)$ in region R tends to the limit l as $x \rightarrow a$ and $y \rightarrow b$ if and only if corresponding to a positive number $\epsilon \in (a, b)$, there exists another positive number δ such that

$$|f(x, y) - l| < \epsilon \text{ for } 0 < (x - a)^2 + (y - b)^2 < \delta^2$$

for every point (x, y) in R .

Continuity : A function $f(x, y)$ is said to be continuous at the point (a, b) if $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = f(a, b)$ irrespective of the path along with $x \rightarrow a$, $y \rightarrow b$.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.1. What is the function of a complex variable ?

Answer

1. If x and y are real variables, then $z = x + iy$ is called a complex variable.
2. If corresponding to each value of a complex variable $z = x + iy$ in a given region R , there correspond one or more values of another complex variable $w = u + iv$, then w is called a function of a complex variable z and is denoted by

$$w = f(z) = u + iv$$

3. For example, if $w = z^2$ where, $z = x + iy$ and $w = f(z) = u + iv$

Then $u + iv = (x + iy)^2 = (x^2 - y^2) + i(2xy)$

$\Rightarrow u = x^2 - y^2$ and $v = 2xy$

4. Thus u and v , the real and imaginary parts of w , are functions of the real variables x and y .

$$w = f(z) = u(x, y) + iv(x, y)$$

5. If to each value of z , there corresponds one and only one value of w , then w is called a single-valued function of z .
6. If to each value of z , there correspond more than one value w , then w is called a multi-valued function of z .

Que 4.2 Evaluate $\lim_{\substack{x \rightarrow 1 \\ y \rightarrow 2}} \frac{3x^2y}{x^2 + y^2 + 5}$.

Answer

$$\begin{aligned} \lim_{\substack{x \rightarrow 1 \\ y \rightarrow 2}} \frac{3x^2y}{x^2 + y^2 + 5} &= \lim_{x \rightarrow 1} \left[\lim_{y \rightarrow 2} \frac{3x^2y}{x^2 + y^2 + 5} \right] = \lim_{x \rightarrow 1} \frac{3x^2(2)}{x^2 + (2)^2 + 5} \\ &= \lim_{x \rightarrow 1} \frac{6x^2}{x^2 + 9} = \frac{6}{1+9} = \frac{3}{5} \end{aligned}$$

Que 4.3 Evaluate $\lim_{\substack{x \rightarrow \infty \\ y \rightarrow 2}} \frac{xy+4}{x^2+2y^2}$.

Answer

$$\begin{aligned} \lim_{\substack{x \rightarrow \infty \\ y \rightarrow 2}} \frac{xy+4}{x^2+2y^2} &= \lim_{x \rightarrow \infty} \left[\lim_{y \rightarrow 2} \frac{xy+4}{x^2+2y^2} \right] = \lim_{x \rightarrow \infty} \left[\frac{x(2)+4}{x^2+2(2)^2} \right] = \lim_{x \rightarrow \infty} \frac{2x+4}{x^2+8} \\ &= \lim_{x \rightarrow \infty} \frac{2 + \frac{4}{x}}{x + \frac{8}{x}} = \frac{2+0}{\infty+0} = 0 \end{aligned}$$

Que 4.4 Show that the function $f(x, y) = x - y$ is continuous for all $(x, y) \in R^2$.

Answer

Let $(a, b) \in R^2$ then $f(a, b) = a - b$

$$\begin{aligned} \therefore |f(x, y) - f(a, b)| &= |(x - y) - (a - b)| \\ &= |(x - a) + (b - y)| \\ &\leq |x - a| + |y - b| \quad [\because |x| = |-x|] \quad \dots(4.4.1) \end{aligned}$$

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{2}$ then for $|x - a| < \delta$ and $|y - b| < \delta$, we have from eq. (4.4.1)

$$|f(x, y) - f(a, b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence, the function $f(x, y) = x - y$ is continuous for all $(a, b) \in R^2$. But (a, b) is an arbitrary element of R^2 , so $f(x, y) = x - y$ is continuous for all $(x, y) \in R^2$.

Que 4.5. If $f(x, y) = \frac{x^3 - y^3}{x^2 + y^2}$ when $x \neq 0, y \neq 0$ and $f(x, y) = 0$ when $x = 0, y = 0$, find out whether the function $f(x, y)$ is continuous at origin.

Answer

First calculate the limit of the function :

- $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^3 - y^3}{x^2 + y^2} = \lim_{y \rightarrow 0} \left(\frac{-y^3}{y^2} \right) = \lim_{y \rightarrow 0} (-y) = 0$
- $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^3 - y^3}{x^2 + y^2} = \lim_{x \rightarrow 0} \left(\frac{x^3}{x^2} \right) = \lim_{x \rightarrow 0} (x) = 0$
- $\lim_{\substack{y \rightarrow \max \\ x \rightarrow 0}} \frac{x^3 - y^3}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{x^3 - m^3 x^3}{x^2 + m^2 x^2} = \lim_{x \rightarrow 0} \frac{(1 - m^3)}{(1 + m^2)} x = 0$
- $\lim_{\substack{y \rightarrow \max \\ x \rightarrow 0}} \frac{x^3 - y^3}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{x^3 - m^3 x^3}{x^2 + m^2 x^2} = \lim_{x \rightarrow 0} \frac{x^3(1 - m^3)}{x^2(1 + m^2)} = \lim_{x \rightarrow 0} \frac{(1 - m^3)}{(1 + m^2)} x = 0$

Since the limit along any path is same, the limit exists and equal to zero which is the value of the function $f(x, y)$ at the origin. Hence, the function f is continuous at the origin.

PART-2

Analytic Functions, Cauchy Riemann Equations (Cartesian and Polar Form).

CONCEPT OUTLINE

Cauchy-Riemann or C-R Equation :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy-Riemann equation in polar coordinates :

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.6. Define analytic function and state the necessary and sufficient condition for function to be analytic.

Answer

A. Analytic function : A function $f(z)$ is said to be analytic at a point z_0 if it is one valued and differentiable not only at z_0 but at every point of some neighbourhood of z_0 .

B. Necessary and sufficient conditions for $f(z)$ to be Analytic : The necessary and sufficient conditions for the function $w = f(z) = u(x, y) + iv(x, y)$ to be analytic in a region R , are

i. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in the region R .

ii. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

The conditions in (ii) are known as Cauchy-Riemann equations or briefly C-R equations.

Que 4.7. Define analytic function. Discuss the analyticity of $f(z) = \operatorname{Re}(z^3)$ in the complex plane. **AKTU 2013-14 (III), Marks 05**

Answer

A. Analytic function : Refer Q. 4.6, Page 4-5 F, Unit-4.

B. Numerical :

Given,

$$\begin{aligned} f(z) &= \operatorname{Re}(z^3) \\ z &= (x + iy) \\ z^3 &= (x + iy)^3 = x^3 - iy^3 + 3xiy(x + iy) \\ &= (x^3 - 3xy^2) + (3x^2y - y^3)i \\ u &= x^3 - 3xy^2 \end{aligned}$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial u}{\partial y} = -6xy$$

$$v = (3x^2y - y^3)$$

$$\frac{\partial v}{\partial x} = 6xy, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$f(z) = \operatorname{Re}(z^3)$ is analytic function.

Que 4.8. Show that $f(z) = \log z$ is analytic everywhere in the complex plane except at the origin. **AKTU 2013-14 (IV), Marks 05**

Answer

Here $f(z) = u + iv = \log z = \log(x + iy)$

($\because z = x + iy$)

Let $x = r \cos \theta$ and $y = r \sin \theta$ so that

$$x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

$$\log(x + iy) = \log(re^{i\theta}) = \log r + i\theta = \frac{1}{2} \log(x^2 + y^2) + i \tan^{-1}\left(\frac{y}{x}\right)$$

Separating real and imaginary parts, we get

$$u = \frac{1}{2} \log(x^2 + y^2) \text{ and } v = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\text{Now, } \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

$$\text{And } \frac{\partial v}{\partial x} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

We observe that the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied except when $x^2 + y^2 = 0$ i.e., when $x = 0, y = 0$

Hence, the function $f(z) = \log z$ is analytic everywhere in the complex plane except at the origin.

Que 4.9. Find the values of c_1 and c_2 such that the function

$$f(z) = x^2 + c_1 y^2 - 2xy + i(c_2 x^2 - y^2 + 2xy)$$

is analytic. Also find $f'(z)$.

AKTU 2016-17 (III), Marks 05

Answer

$$\begin{aligned} f(z) &= x^2 + c_1 y^2 - 2xy + i(c_2 x^2 - y^2 + 2xy) \\ u + iv &= x^2 + c_1 y^2 - 2xy + i(c_2 x^2 - y^2 + 2xy) \end{aligned}$$

Comparing real and imaginary parts, we get

$$u = x^2 + c_1 y^2 - 2xy$$

$$v = c_2 x^2 - y^2 + 2xy$$

And

$$\frac{\partial u}{\partial x} = 2x - 2y \text{ and } \frac{\partial v}{\partial x} = 2c_2 x + 2y$$

$$\frac{\partial u}{\partial y} = 2c_1 y - 2x \text{ and } \frac{\partial v}{\partial y} = -2y + 2x$$

C-R equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$2x - 2y = -2y + 2x \quad \dots(4.9.1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$2c_1y - 2x = -2c_2x - 2y \quad \dots(4.9.2)$$

From eq. (4.9.2), equating the coefficient of x and y , we get

$$2c_1 = -2 \Rightarrow c_1 = -1$$

$$-2 = -2c_2 \Rightarrow c_2 = 1$$

$$\begin{aligned} \text{Now, } f'(z) &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = (2x - 2y) + i(c_2 2x + 2y) \\ &= (2x - 2y) + i(2x + 2y) \quad (\because c_2 = 1) \\ &= 2(x + iy) + 2i(x + iy) \\ &= 2z + 2iz = 2(1 + i)z \end{aligned}$$

Que 4.10. Find p such that the function $f(z)$ expressed in polar coordinates as $f(z) = r^2 \cos 2\theta + ir^2 \sin p\theta$ is analytic.

Answer

Let $f(z) = u + iv$, then $u = r^2 \cos 2\theta$, $v = r^2 \sin p\theta$

$$\frac{\partial u}{\partial r} = 2r \cos 2\theta, \quad \frac{\partial v}{\partial r} = 2r \sin p\theta$$

$$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta, \quad \frac{\partial v}{\partial \theta} = pr^2 \cos p\theta$$

For $f(z)$ to be analytic, $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$

$$2r \cos 2\theta = pr \cos p\theta \text{ and } 2r \sin p\theta = 2r \sin 2\theta$$

Both these equations are satisfied if $p = 2$.

Que 4.11. Show that the function defined by $f(z) = \sqrt{|xy|}$ is not regular at the origin, although Cauchy-Riemann equations are satisfied.

AKTU 2016-17 (IV), Marks 05

Answer

Let

$$f(z) = u(x, y) + iv(x, y)$$

$$= \sqrt{|xy|} \text{ then } u(x, y) = \sqrt{|xy|}, v(x, y) = 0$$

At the origin $(0, 0)$, we have

$$\frac{\partial u}{\partial x} = \lim_{x \rightarrow 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \rightarrow 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \rightarrow 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \rightarrow 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{0 - 0}{y} = 0$$

Clearly,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, C-R equations are satisfied at the origin.

$$\text{Now } f'(0) = \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z} = \lim_{z \rightarrow 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

If $z \rightarrow 0$ along the line $y = mx$, we get

$$f'(0) = \lim_{x \rightarrow 0} \frac{\sqrt{|mx^2|}}{x(1 + im)} = \lim_{x \rightarrow 0} \frac{\sqrt{|m|}}{1 + im}$$

Now this limit is not unique since it depends on m . Therefore, $f'(0)$ does not exist.

Hence, the function $f(z)$ is not regular at the origin.

Que 4.12. Prove that the function $\sinh z$ is analytic and find its derivation.

Answer

Here

$$f(z) = u + iv = \sinh z = \sinh(x + iy) = \sinh x \cosh y + i \cosh x \sinh y$$

$$u = \sinh x \cosh y \text{ and } v = \cosh x \sinh y \quad (\because z = x + iy)$$

$$\frac{\partial u}{\partial x} = \cosh x \cosh y, \quad \frac{\partial u}{\partial y} = \sinh x \sinh y$$

$$\frac{\partial v}{\partial x} = \sinh x \sinh y, \quad \frac{\partial v}{\partial y} = \cosh x \cosh y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus C-R equations are satisfied.

Since $\sinh x$, $\cosh x$, $\sin y$ and $\cos y$ are continuous function, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$

and $\frac{\partial v}{\partial y}$ are also continuous functions satisfying C-R equations.

Hence $f(z)$ is analytic everywhere.

$$\begin{aligned} \text{Now, } f'(z) &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\ &= \cosh x \cosh y + i \sinh x \sinh y \\ &= \cosh(x + iy) = \cosh z \end{aligned}$$

Que 4.13. Using C - R equations show that $f(z) = |z|^2$ is not analytical at any point.

AKTU 2014-15 (IV), Marks 05

Answer

Let $w = f(z) = u + iv = |z|^2$
 $u + iv = x^2 + y^2$

Comparing both sides,

$$u = x^2 + y^2, \quad \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y$$

$$v = 0, \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

Using C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2x = 0 \Rightarrow x = 0$$

And $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 2y = 0 \Rightarrow y = 0$

At (0, 0) C-R equations are satisfied and the function is differentiable.
Hence, the function is not analytic anywhere except at origin.

Que 4.14. If $f(z) = \frac{x^3 y(y - ix)}{x^6 + y^2}$ when $z \neq 0$
 $= 0$ when $z = 0$

Prove that $\frac{f(z) - f(0)}{z} \rightarrow 0$ as $z \rightarrow 0$ along any radius vector but not as $z \rightarrow 0$ in any manner. **AKTU 2012-13 (III), Marks 05**

Answer

$$f(z) = u + iv = \frac{x^3 y(y - ix)}{x^6 + y^2}, \quad z \neq 0$$

$$u = \frac{x^3 y^2}{x^6 + y^2}, \quad v = \frac{-x^4 y}{x^6 + y^2}$$

$$\frac{\partial u}{\partial x} = \lim_{h \rightarrow 0} \frac{u(0+h, 0) - u(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0/h^3}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \rightarrow 0} \frac{u(0, 0+k) - u(0, 0)}{k} = \lim_{k \rightarrow 0} \frac{0/k^2}{k} = 0$$

Similarly, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence, C-R equations are satisfied at origin.

Now, $\frac{f(z) - f(0)}{z} = \left[\frac{x^3 y(y - ix)}{x^6 + y^2} - 0 \right] \frac{1}{x + iy}$
 $= \frac{x^3 y(y - ix)}{x^6 + y^2} \frac{1}{(x + iy)} = \frac{-ix^3 y}{x^6 + y^2}$

Let $z \rightarrow 0$ along radius vector $y = mx$, then

$$\lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z} = \lim_{x \rightarrow 0} \frac{-ix^3(mx)}{x^6 + m^2 x^2} = \lim_{x \rightarrow 0} \frac{-imx^2}{x^4 + m^2} = 0$$

Hence $\frac{f(z) - f(0)}{z} \rightarrow 0$ as $z \rightarrow 0$ along any radius vector.

Let $z \rightarrow 0$ along $y = x^3$ then,

$$\lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z} = \lim_{x \rightarrow 0} \frac{-ix^3 x^3}{x^6 + x^6} = \frac{-i}{2}$$

Thus $f'(0)$ does not exist, hence $f(z)$ is not analytic at $z = 0$.

Que 4.15. Let $f(z) = \frac{x^3 y^5(x + iy)}{x^4 + y^{10}}$ when $z \neq 0$, $f(z) = 0$ when $z = 0$. Prove

that Cauchy-Riemann satisfies at $z = 0$ but function is not differentiable at $z = 0$. **AKTU 2018-19, Marks 10**

OR

Examine the nature of the function

$$f(z) = \frac{x^3 y^5(x + iy)}{x^4 + y^{10}}; \quad z \neq 0, \quad f(0) = 0$$

in the region including the origin. **AKTU 2015-16 (III), Marks 10**

Answer

Here, $u + iv = \frac{x^3 y^5(x + iy)}{x^4 + y^{10}}; \quad z \neq 0$

$$u = \frac{x^3 y^5}{x^4 + y^{10}}, \quad v = \frac{x^3 y^6}{x^4 + y^{10}}$$

At the origin, $\frac{\partial u}{\partial x} = \lim_{x \rightarrow 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{0 - 0}{x} = 0$

$$\frac{\partial u}{\partial y} = \lim_{y \rightarrow 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{0 - 0}{y} = 0$$

Similarly, $\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y}$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence Cauchy-Riemann equations are satisfied at the origin

$$\begin{aligned} \text{But } f'(0) &= \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z} = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[\frac{x^2 y^5 (x + iy)}{x^4 + y^{10}} - 0 \right] \frac{1}{x + iy} \\ &= \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 y^5}{x^4 + y^{10}} \end{aligned}$$

Let $z \rightarrow 0$ along the radius vector $y = mx$, then

$$f'(0) = \lim_{x \rightarrow 0} \frac{m^5 x^7}{x^4 + m^{10} x^{10}} = \lim_{x \rightarrow 0} \frac{m^5 x^3}{1 + m^{10} x^6} = 0$$

Again let $z \rightarrow 0$ along the curve $y^5 = x^2$

$$f'(0) = \lim_{x \rightarrow 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2}$$

which shows that $f'(0)$ does not exist. Hence $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there.

Que 4.16. Prove that the function $f(z)$ defined by

$$f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist. **AKTU 2016-17 (IV), Marks 10**

Answer

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} = \frac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2} = u + iv$$

$$\text{Where, } u = \frac{x^3 - y^3}{x^2 + y^2}, v = \frac{x^3 + y^3}{x^2 + y^2}$$

$\therefore$ The value of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at $(0, 0)$ we get $\frac{0}{0}$, so we apply first principle method.
At the origin,

$$\frac{\partial u}{\partial x} = \lim_{h \rightarrow 0} \frac{u(0+h, 0) - u(0, 0)}{h} = \lim_{h \rightarrow 0} \left(\frac{h^3}{h^2} \right) / h = 1$$

$$\frac{\partial u}{\partial y} = \lim_{k \rightarrow 0} \frac{u(0, 0+k) - u(0, 0)}{k} = \lim_{k \rightarrow 0} \left(\frac{-k^3}{k^2} \right) / k = -1$$

$$\frac{\partial v}{\partial x} = \lim_{h \rightarrow 0} \frac{v(0+h, 0) - v(0, 0)}{h} = \lim_{h \rightarrow 0} \left(\frac{h^3}{h^2} \right) / h = 1$$

$$\frac{\partial v}{\partial y} = \lim_{k \rightarrow 0} \frac{v(0, 0+k) - v(0, 0)}{k} = \lim_{k \rightarrow 0} \left(\frac{k^3}{k^2} \right) / k = 1$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, C-R equations are satisfied at origin.

$$\text{Now } f'(0) = \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z} = \lim_{z \rightarrow 0} \left[\frac{x^3 - y^3 + i(x^3 + y^3)}{x^2 + y^2} \frac{1}{x + iy} \right]$$

Now let $z \rightarrow 0$ along $y = mx$, then

$$\begin{aligned} f'(0) &= \lim_{x \rightarrow 0} \left[\frac{x^3 - m^3 x^3 + i(x^3 + m^3 x^3)}{x^2 + m^2 x^2} \frac{1}{x + imx} \right] \\ &= \lim_{x \rightarrow 0} \left[\frac{1 - m^3 + i(1 + m^3)}{(1 + m^2)(1 + im)} \right] = \frac{m^3(-1+i) + (1+i)}{(1 + m^2)(1 + im)} \end{aligned}$$

$\therefore$ The value of $f'(0)$ depends on m , therefore $f'(0)$ is not unique.
Hence, the function is not analytic at $z = 0$.

PART-3

Harmonic Function, Method to Find Analytic Functions, Milne's Thomson Method.

CONCEPT OUTLINE

Harmonic Function : A function of (x, y) which possesses continuous partial derivatives of the first and second orders and satisfies Laplace equation is called a harmonic function.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(1)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \dots(2)$$

Equations (1) and (2) shows that the real and imaginary parts u and v of an analytic function satisfy the Laplace equation. Hence u and v are harmonic functions.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.17. Verify that the function on $u(x, y) = xy$ is harmonic and find its conjugate harmonic function. Express $u + iv$ as an analytic function $f(z)$.

$$u = x^2 - y^2 - y$$

AKTU 2015-16 (III), Marks 05

Answer

$$u(x, y) = xy$$

$$\frac{\partial u}{\partial x} = y \quad \therefore \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial y} = x \quad \therefore \frac{\partial^2 u}{\partial y^2} = 0$$

For a function to be harmonic, it must satisfy Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence, function $u(x, y)$ is harmonic.

Using Cauchy-Riemann equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Total differentiation of v is given as,

$$\text{On integrating, } dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = -x dx + y dy$$

$$v = \frac{-x^2}{2} + \frac{y^2}{2} + c$$

u and v are said complex conjugate.

Again,

$$u = x^2 - y^2 - y$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y - 1$$

Using Cauchy-Riemann equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = (2y + 1) dx + 2x dy = d(2xy + x)$$

$$\therefore v = 2xy + x + c$$

Then,

$$f(x, y) = u + iv = (x^2 - y^2 - y) + i(2xy + x + c)$$

Que 4.18. Show that $v(x, y) = e^{-x} (x \cos y + y \sin y)$ is harmonic. Find its harmonic conjugate.

AKTU 2013-14 (III), Marks 05

Answer

$$v(x, y) = e^{-x} (x \cos y + y \sin y)$$

$$\frac{\partial v}{\partial x} = -e^{-x} (x \cos y + y \sin y) + e^{-x} (\cos y)$$

$$\frac{\partial v}{\partial y} = e^{-x} (-x \sin y + y \cos y + \sin y)$$

$$\frac{\partial^2 v}{\partial x^2} = -[-e^{-x} (x \cos y + y \sin y) + e^{-x} (\cos y)] - e^{-x} (\cos y)$$

$$\frac{\partial^2 v}{\partial y^2} = e^{-x} [-x \cos y + (\cos y - y \sin y) + \cos y]$$

$$= e^{-x} [2 \cos y - y \sin y - x \cos y]$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^{-x} [x \cos y + y \sin y - \cos y - \cos y]$$

$$+ e^{-x} [2 \cos y - y \sin y - x \cos y]$$

$$= e^{-x} [x \cos y + y \sin y - 2 \cos y + 2 \cos y - y \sin y - x \cos y]$$

$$= 0$$

Since, v satisfies the Laplace equation hence v is harmonic function.

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \left(\frac{\partial v}{\partial y} \right) dx + \left(-\frac{\partial v}{\partial x} \right) dy$$

$$du = [e^{-x} (-x \sin y + y \cos y + \sin y)] dx + e^{-x} [x \cos y + y \sin y - \cos y] dy$$

$$u = \int_{y=\text{constant}} e^{-x} (-x \sin y + y \cos y + \sin y) dx + \int_{x=\text{constant}} e^{-x} (x \cos y + y \sin y - \cos y) dy$$

$$u = -\int e^{-x} x \sin y dx + y \cos y \int e^{-x} dx + \sin y \int e^{-x} dx$$

$$+ x e^{-x} \int \cos y dy + e^{-x} \int y \sin y dy - e^{-x} \int \cos y dy$$

$$u = -(-2x e^{-x}) \sin y - e^{-x} y \cos y - e^{-x} \sin y + x e^{-x} \sin y + e^{-x} (-y \cos y - y \sin y) - e^{-x} \sin y$$

$$u = 2x e^{-x} \sin y - e^{-x} y \cos y - e^{-x} \sin y + x e^{-x} \sin y$$

$$- e^{-x} y \cos y - e^{-x} y \sin y - e^{-x} \sin y$$

$$u = 3x e^{-x} \sin y - 2e^{-x} y \cos y - e^{-x} y \sin y - 2e^{-x} \sin y$$

Here u is the harmonic conjugate of v .

Que 4.19. Show that the function $u = \frac{1}{2} \log(x^2 + y^2)$ is harmonic.

Find the harmonic conjugate of u . **AKTU 2014-15 (III), Marks 05**

Answer

$$u = \frac{1}{2} \log(x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \text{ hence } u \text{ is harmonic.}$$

$$\text{Now, } dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad (\text{By C-R equations})$$

$$dv = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

$$dv = \frac{x dy - y dx}{x^2 + y^2} = d \left[\tan^{-1} \left(\frac{y}{x} \right) \right]$$

Integration yields, $v = \tan^{-1} \left(\frac{y}{x} \right) + c$

This is the required harmonic conjugate function of u .

Que 4.20. Discuss about the Milne's Thomson method.

Answer

With the help of Milne's Thomson method, we can directly construct $f(z)$ in terms of z without finding out v when u is given or u when v is given.

$$z = x + iy$$

$$\bar{z} = x - iy$$

$$\Rightarrow x = \frac{1}{2}(z + \bar{z}) \text{ and } y = \frac{1}{2i}(z - \bar{z})$$

$$\therefore f(z) = u(x, y) + iv(x, y)$$

$$= u \left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right) + iv \left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right) \quad \dots(4.20.1)$$

Relation (4.20.1) is an identity in z and $\bar{z}$. Putting $\bar{z} = z$, we get

$$f(z) = u(z, 0) + iv(z, 0) \quad (4.20.2)$$

$$\text{Now, } f(z) = u + iv$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \quad [\text{By C-R equation}]$$

$$= \phi_1(x, y) - i \phi_2(x, y)$$

$$\left[\text{where, } \phi_1(x, y) = \frac{\partial u}{\partial x} \text{ and } \phi_2(x, y) = \frac{\partial u}{\partial y} \right]$$

$$\text{Now, } f'(z) = \phi_1(z, 0) - i \phi_2(z, 0) \quad [\text{Replacing } x \text{ by } z \text{ and } y \text{ by } 0]$$

Integrating, we get

$$f(z) = \int (\phi_1(z, 0) - i \phi_2(z, 0)) dz + c$$

Hence the function is constructed directly in terms of z .

Similarly if $v(x, y)$ is given, then

$$f(z) = \int [\psi_1(z, 0) + i \psi_2(z, 0)] dz + c$$

$$\left[\text{where, } \psi_1(x, y) = \frac{\partial v}{\partial y} \text{ and } \psi_2(x, y) = \frac{\partial v}{\partial x} \right]$$

Milne's Thomson method can easily be grasped by going through the steps involved in following various cases :

Case I : When only real part $u(x, y)$ is given.

To construct analytic function $f(z)$ directly in terms of z when only real part u is given, we use the following steps :

1. Find $\frac{\partial u}{\partial x}$
2. Write it as equal to $\phi_1(x, y)$
3. Find $\frac{\partial u}{\partial y}$
4. Write it as equal to $\phi_2(x, y)$
5. Find $\phi_1(z, 0)$ by replacing x by z and y by 0 in $\phi_1(x, y)$.
6. Find $\phi_2(z, 0)$ by replacing x by z and y by 0 in $\phi_2(x, y)$.
7. $f(z)$ is obtained by the formula

$$f(z) = \int (\phi_1(z, 0) + i \phi_2(z, 0)) dz + c \text{ directly in terms of } z.$$

Case II : When only imaginary part $v(x, y)$ is given.

To construct analytic function $f(z)$ directly in terms of z when only imaginary part v is given, we use the following steps :

- Find $\frac{\partial v}{\partial y}$
- Write it as equal to $\psi_1(x, y)$
- Find $\frac{\partial v}{\partial x}$
- Write it as equal to $\psi_2(x, y)$
- Find $\psi_1(z, 0)$ by replacing x by z and y by 0 in $\psi_1(x, y)$
- Find $\psi_2(z, 0)$ by replacing x by z and y by 0 in $\psi_2(x, y)$
- $f(z)$ is obtained by the formula

$$f(z) = \int \{\psi_1(z, 0) + i\psi_2(z, 0)\} dz + c \text{ directly in terms of } z.$$

Case III : When $u - v$ is given.

To construct analytic function $f(z)$ directly in terms of z when $u - v$ is given, we follow the following steps :

- $f(z) = u + iv$... (4.20.3)
- $i f(z) = iu - v$... (4.20.4)
- Add eq. (4.20.3) and (4.20.4) to get
 $(1 + i) f(z) = (u - v) + i(u + v)$
 or, $F(z) = U + iV$
 where, $F(z) = (1 + i)f(z)$, $U = u - v$ and $V = u + v$
- Since $u - v$ is given, hence $U(x, y)$ is given
- Find $\frac{\partial U}{\partial x}$
- Write it as equal to $\phi_1(x, y)$
- Find $\frac{\partial U}{\partial y}$
- Write it as equal to $\phi_2(x, y)$
- Find $\phi_1(z, 0)$
- Find $\phi_2(z, 0)$
- $F(z)$ is obtained by the formula,

$$F(z) = \int \{\phi_1(z, 0) - i\phi_2(z, 0)\} dz + c$$

- $f(z)$ is determined by $f(z) = \frac{F(z)}{1 + i}$ directly in terms of z .

Case IV : When $u + v$ is given.

To construct analytic function $f(z)$ directly in terms of z when $u + v$ is given, we follow the following steps :

- $f(z) = u + iv$... (4.20.5)
- $if(z) = iu - v$... (4.20.6)
- Add eq. (4.20.5) and (4.20.6) to get
 $(1 + i) f(z) = (u - v) + i(u + v)$
 $\Rightarrow F(z) = U + iV$
 where, $F(z) = (1 + i)f(z)$, $U = u - v$ and $V = u + v$
- Since $u + v$ is given hence $V(x, y)$ is given
- Find $\frac{\partial V}{\partial y}$
- Write it as equal to $\psi_1(x, y)$
- Find $\frac{\partial V}{\partial x}$
- Write it as equal to $\psi_2(x, y)$
- Find $\psi_1(z, 0)$
- Find $\psi_2(z, 0)$
- $F(z)$ is obtained by the formula

$$F(z) = \int \{\psi_1(z, 0) + i\psi_2(z, 0)\} dz + c$$

- $f(z)$ is determined by $f(z) = \frac{F(z)}{1 + i}$ directly in terms of z .

Que 4.21. Find the analytic function whose real part is

$$e^{2x} (x \cos 2y - y \sin 2y).$$

AKTU 2014-15 (IV), Marks 05

Answer

Let, $u = e^{2x} (x \cos 2y - y \sin 2y)$

$$\frac{\partial u}{\partial x} = e^{2x} (\cos 2y) + 2e^{2x} (x \cos 2y - y \sin 2y) = \phi_1(x, y)$$

$$\frac{\partial u}{\partial y} = e^{2x} [-2x \sin 2y - 2y \cos 2y - \sin 2y] = \phi_2(x, y)$$

Now, $\phi_1(z, 0) = e^{2x}(2x + 1)$

$$\phi_2(z, 0) = e^{-2x}(0) = 0$$

By Milne's Thomson method,

$$f(z) = \int \{\phi_1(z, 0) - i\phi_2(z, 0)\} dz + c = \int e^{2z}(2z + 1) dz + c$$

$$= (2z + 1) \frac{e^{2z}}{2} - \int 2 \frac{e^{2z}}{2} dz + c \quad (\because \text{Integrating by parts})$$

$$= (2z+1)\frac{e^{2z}}{2} - \frac{1}{2}e^{2z} + x$$

$$f(z) = ze^{2z} + c$$

Que 4.22. Find an analytic function whose imaginary part is $e^{-x}(x \cos y + y \sin y)$. **AKTU 2013-14 (IV), Marks 05**

Answer

Let $f(z) = u + iv$ be the required analytic function.

Here, $v = e^{-x}(x \cos y + y \sin y)$

$$\frac{\partial v}{\partial y} = e^{-x}(-x \sin y + y \cos y + \sin y) = \psi_1(x, y)$$

$$\frac{\partial v}{\partial x} = e^{-x} \cos y - e^{-x}(x \cos y + y \sin y) = \psi_2(x, y)$$

$$\psi_1(z, 0) = 0, \psi_2(z, 0) = e^{-z} - e^{-z}z = (1-z)e^{-z}$$

By Milne's Thomson method,

$$f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)] dz + c = i \int (1-z)e^{-z} dz + c$$

$$= i \left[(1-z)(-e^{-z}) - \int (-1)(-e^{-z}) dz \right] + c$$

$$= i[(z-1)e^{-z} + e^{-z}] + c$$

$$f(z) = iz e^{-z} + c$$

Que 4.23. Show that $u = x^4 - 6x^2y^2 + y^4$ is harmonic function. Find complex function $f(z)$ whose u is a real part. **AKTU 2013-14, Marks 10**

Answer

$$u = x^4 - 6x^2y^2 + y^4$$

$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$$

$$\frac{\partial u}{\partial y} = -12x^2y + 4y^3 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2$$

Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \therefore u(x, y)$ is a harmonic function.

Now, let $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$$= \left(-\frac{du}{dy}\right) dx + \frac{\partial u}{\partial x} dy \quad (\text{By C-R equation})$$

$$= (12x^2y - 4y^3) dx + (4x^3 - 12xy^2) dy$$

$$= (12x^2y dx + 4x^3 dy) - (4y^3 dx + 12xy^2 dy)$$

$$= d(4x^3y) - d(4xy^3)$$

Integration yields, $v = 4x^3y - 4xy^3 + c$

Hence $f(z) = u + iv = x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3 + c)$

$$= (x + iy)^4 + c_1 = z^4 + c_1 \quad (\text{Where, } c_1 = ic)$$

Que 4.24. If $u = 3x^2y - y^3$ find the analytic function $f(z) = u + iv$. **AKTU 2012-13 (III), Marks 05**

Answer

$$u = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 6xy, \quad \frac{\partial u}{\partial y} = 3x^2 - 3y^2$$

Now,

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$$

$$dv = -(3x^2 - 3y^2)dx + 6xy dy$$

On integrating,

$$v = \int M dx + \int N dy$$

(y as constant) (ignoring terms of x)

$$v = \int (3y^2 - 3x^2) dx + 0 = 3xy^2 - x^3 + c$$

Now,

$$\begin{aligned} u + iv &= 3x^2y - y^3 + i(3xy^2 - x^3 + c) \\ &= (3x^2y - y^3 + i(3xy^2 - x^3)) + ic \\ &= -i(x^2 - iy^3) + 3xy^2 + 3ix^2y + ic \\ &= -i(x + iy)^3 + ic = -iz^3 + ic \end{aligned}$$

$$u + iv = -i(z^3 - c)$$

Que 4.25. If $f(z) = u + iv$ is analytic function and $u - v = e^x(\cos y - \sin y)$, find $f(z)$ in terms of z . **AKTU 2015-16 (III), Marks 05**

Answer

$$u + iv = f(z) \quad \dots(4.25.1)$$

$$i(u + iv) = if(z)$$

$$iu - v = if(z) \quad \dots(4.25.2)$$

On adding eq. (4.25.1) and eq. (4.25.2),

$$u - v + i(u + v) = (1 + i)f(z)$$

$$U + iV = F(z)$$

Where, $U = u - v = e^x (\cos y - \sin y)$
 $V = u + v$

$$(1 + i)f(z) = F(z)$$

Now using Milne's Thomson method,

$$\frac{\partial U}{\partial x} = \phi_1 = e^x (\cos y - \sin y)$$

So, $\phi_1(z, 0) = e^z (\cos 0 - \sin 0)$
 $\phi_1(z, 0) = e^z$

Now $\frac{\partial U}{\partial y} = \phi_2 = e^x (-\sin y - \cos y)$

$$\phi_2(z, 0) = e^z (-\sin 0 - \cos 0)$$

$$\phi_2(z, 0) = -e^z$$

According to Milne's Thomson method,

$$F(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)] dz + c$$

$$= \int (e^z + ie^z) dz + c = \int e^z (1 + i) dz + c$$

$$F(z) = (1 + i)e^z + c$$

or $(1 + i)f(z) = (1 + i)e^z + c$

$$f(z) = e^z + \frac{c}{1 + i}$$

Que 4.26. If $f(z) = u + iv$ is an analytic function, find $f(z)$ in term of

$$z \text{ if } u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x} \text{ when } f\left(\frac{\pi}{2}\right) = \frac{3 - i}{2}.$$

AKTU 2021-22, Marks 10

Answer

We have, $u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{\cosh y + \sinh y - \cos x + \sin x}{\cosh y - \cos x}$$

$$= \frac{\cosh y - \cos x}{\cosh y - \cos x} + \frac{\sinh y + \sin x}{\cosh y - \cos x}$$

$$= 1 + \frac{\sinh y + \sin x}{\cosh y - \cos x}$$

Differentiate wrt x partially, we get

$$\frac{\partial u}{\partial x} = \frac{-[(\cos x - \cosh y) \cos x - (\sin x + \sinh y)(-\sin x)]}{(\cos x - \cosh y)^2}$$

$$\phi(x, y) = -1 \times \frac{(1 - \cosh y \cos x + \sinh y \sin x)}{(\cos x - \cosh y)^2}$$

$$\phi(z, 0) = \frac{-(1 - \cos z)}{(\cos z - 1)^2} = \frac{-1}{1 - \cos z}$$

Differentiate wrt y partially, we get

$$\frac{\partial v}{\partial y} = \frac{-[(\cos x - \cosh y) \cosh y - (\sin x + \sinh y)(-\sinh y)]}{(\cos x - \cosh y)^2}$$

$$\phi_2(x, y) = \frac{-[(\cos x - \cosh y) + \sinh y \sinh y - 1]}{(\cos x - \cosh y)^2}$$

$$\phi_2(z, 0) = \frac{-(\cos z - 1)}{(\cos z - 1)^2} = \frac{1}{1 - \cos z}$$

By Milne Thomson method,

$$F(z) = \int [\phi(z, 0) - i\phi_2(z, 0)] dz + C \quad \dots(4.26.1)$$

$$= \int \left[\frac{1}{1 - \cos z} - \frac{i}{1 - \cos z} \right] dz + C$$

$$= -(1 + i) \int \frac{1}{2 \sin^2\left(\frac{z}{2}\right)} dz + C$$

$$= -\frac{(1 + i)}{2} \operatorname{cosec}^2\left(\frac{z}{2}\right) dz + C$$

$$= -\frac{(1 + i)}{2} \frac{(-\cot \frac{z}{2})}{1/2}$$

$$= (1 + i) \cot\left(\frac{z}{2}\right) dz + C$$

$$f(z) = \cot \frac{z}{2} + C$$

Put $z = \frac{\pi}{2}$, we get

$$f\left(\frac{\pi}{2}\right) = \frac{3-i}{2} = \cot \frac{\pi}{4} + C \quad \dots(4.26.2)$$

$$\Rightarrow C = \frac{1-i}{2}$$

Putting the value of C in eq. (4.26.2), we get

$$f(z) = \cot \frac{z}{2} + \left(\frac{1-i}{2}\right)$$

Que 4.27. Determine an analytic function $f(z)$ in term of z if

$$u + v = 2 \frac{\sin 2x}{e^{2y}} + e^{2y} - 2 \cos 2x.$$

AKTU 2017-18 (IV), Marks 07

Answer

Let

$$f(z) = u + iv$$

$$if(z) = iu - v$$

$$(1+i)f(z) = (u-v) + i(u+v)$$

$$F(z) = U + iV$$

$$U = (u-v) \text{ and } V = u+v$$

Where,

$$\text{Hence, } V = u+v = \frac{2 \sin 2x}{e^{2y}} + e^{2y} - 2 \cos 2x$$

$$\text{Now, } \frac{\partial V}{\partial x} = \frac{4 \cos 2x}{e^{2y}} + 4 \sin 2x = \psi_1(x, y)$$

$$\text{and } \frac{\partial V}{\partial y} = \frac{-4 \sin 2x}{e^{2y}} + 2e^{2y} = \psi_2(x, y)$$

$$\therefore \psi_1(z, 0) = -4 \sin 2x + 2$$

$$\psi_2(z, 0) = 4 \cos 2x + 4 \sin 2x$$

By Milne's Thomson method,

$$\begin{aligned} F(z) &= \int (\psi_1(z, 0) + i \psi_2(z, 0)) dz + c \\ &= \int ((-4 \sin 2z + 2) + i(4 \cos 2z + 4 \sin 2z)) dz + c \\ &= \left(\frac{4 \cos 2z}{2} + 2z \right) + i \left(\frac{4 \sin 2z}{2} - \frac{4 \cos 2z}{2} \right) + c \\ &= (2 \cos 2z + 2z) + i(2 \sin 2z - 2 \cos 2z) + c \end{aligned}$$

$$\text{or } (1+i)f(z) = (2 \cos 2z + 2z) + i(2 \sin 2z - 2 \cos 2z) + c$$

$$\text{or } f(z) = \frac{2(\cos 2z + z)}{(1+i)} + \frac{2i(\sin 2z - \cos 2z)}{(1+i)} + \frac{c}{(1+i)}$$

Multiply and divide by $(1-i)$ on RHS, we get

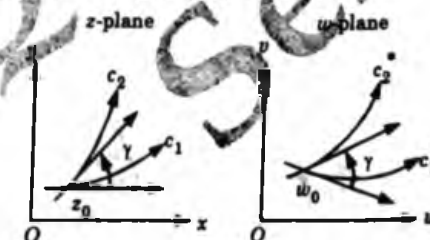
$$\begin{aligned} f(z) &= \frac{2(\cos 2z + z)}{(1+i)} \left(\frac{1-i}{1-i} \right) \\ &\quad + \frac{2i(\sin 2z - \cos 2z)}{(1+i)} \left(\frac{1-i}{1-i} \right) + \frac{c}{(1+i)} \left(\frac{1-i}{1-i} \right) \\ &= \frac{2(1-i)(z + \cos 2z)}{1^2 - i^2} + \frac{2i(1-i)(\sin 2z - \cos 2z)}{1^2 - i^2} + c_1 \\ &\quad \text{(Where, } c_1 = \text{Constant)} \\ &= \frac{2(1-i)(z + \cos 2z)}{2} + \frac{2i(1-i)(\sin 2z - \cos 2z)}{2} + c_1 \quad (\because i^2 = -1) \\ &= (z + \cos 2z) - i(z + \cos 2z) + (i+1)(\sin 2z - \cos 2z) \\ &= (z + \cos 2z + \sin 2z - \cos 2z) + i(-z - \cos 2z + \sin 2z - \cos 2z) \\ f(z) &= (z + \sin 2z) + i(\sin 2z - 2 \cos 2z - z) \end{aligned}$$

PART-4

Conformal Mapping.

CONCEPT OUTLINE

Conformal Mapping: A mapping $w = f(z)$ is said to be conformal if the angle between any two smooth curves c_1, c_2 in the z -plane intersecting at the point z_0 is equal in magnitude and sense to the angle between their images c_1', c_2' in the w -plane at the point $w_0 = f(z_0)$.



General Linear Transformation: General linear transformation or simply linear transformation defined by the function

$$w = f(z) = az + b \quad \dots(1)$$

($a \neq 0$, and b are arbitrary complex constants) maps conformally the extended complex z -plane onto the extended w -plane, since this function is analytic and $f'(z) = a \neq 0$ for any z . If $a = 0$, eq. (1) reduces to a constant function.

Special Cases of Linear Transformation :

- i. **Identity Transformation :** In this, $w = z$ for $a = 1, b = 0$, which maps a point z onto itself.
- ii. **Translation :** In this, $w = z + b$ for $a = 1$, which translates (shifts) z through a distance $|b|$ in the direction of b .
- iii. **Rotation :** In this, $w = e^{i\theta_0} z$ for $a = e^{i\theta_0}, b = 0$ which rotates (the radius vector of point) z through a scalar angle θ_0 (counterclockwise if $\theta_0 > 0$, while clockwise of $\theta_0 < 0$).
- iv. **Stretching (Scaling) :** In this, $w = az$ for 'a' real stretches if $a > 1$ (contracts if $0 < a < 1$) the radius vector by a factor 'a'.

Questions-Answers**Long Answer Type and Medium Answer Type Questions**

Que 4.28. State and prove condition for conformality.

Answer

Statement : A mapping $w = f(z)$ is conformal at each point z_0 where $f(z)$ is analytic and $f'(z_0) \neq 0$.

Proof : Since f is analytic, f' exists and since $f' \neq 0$, we have at a point z_0

$$R_0 e^{i\theta_0} = f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \left(\left| \frac{\Delta w}{\Delta z} \right| + i \arg \frac{\Delta w}{\Delta z} \right)$$

$$\theta_0 = \lim_{\Delta z \rightarrow 0} \left(\arg \frac{\Delta w}{\Delta z} \right)$$

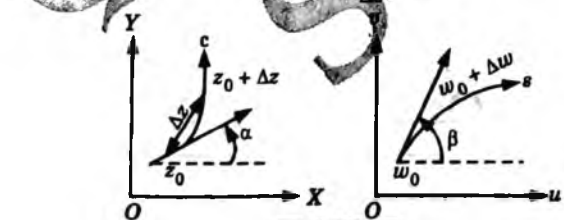


Fig. 4.28.1

Since

$$\Delta w = \frac{\Delta w}{\Delta z} \Delta z$$

$$\arg \Delta w = \arg \frac{\Delta w}{\Delta z} + \arg \Delta z$$

As $\Delta z \rightarrow 0$, $\beta = \theta_0 + \alpha$

Thus the directed tangent to curve c at z_0 is rotated through an angle $\theta_0 = \arg f'(z_0)$, which is same for all curves through z_0 . Let α_1, α_2 be angles of inclination of two curves c_1 and c_2 and β_1 and β_2 be the corresponding angles for their images S_1 and S_2 .

Then $\beta_1 = \alpha_1 + \theta_0$ and $\beta_2 = \alpha_2 + \theta_0$

Thus $\beta_2 - \beta_1 = \alpha_2 - \alpha_1 = \gamma$

Hence, the angle γ between the curves c_1 and c_2 and their images S_1 and S_2 is same both in magnitude and sense.

Que 4.29. Show that circles are invariant under translation, rotation and stretching.

Answer

Linear transformation preserves circles i.e., a circle in the z -plane under linear transformation maps to a circle in the w -plane.

Consider any circle in the z -plane

$$A(x^2 + y^2) + Bx + Cy + D = 0 \quad \dots(4.29.1)$$

Let $w = f(z) = az + b$

From eq. (4.29.1), $u + iv = w = az + b = a(x + iy) + (b_1 + ib_2)$

or $u = ax + b_1, v = ay + b_2$

$$\text{or } x = \frac{u - b_1}{a}, y = \frac{v - b_2}{a}, a \neq 0 \quad \dots(4.29.2)$$

Substituting the value of x and y from eq. (4.29.2) in eq. (4.29.1), we get

$$A^*(u^2 + v^2) + B^*u + C^*v + D^* = 0 \quad \dots(4.29.3)$$

Which is circle in the w -plane.

Where,

$$A^* = \frac{A}{a^2}, B^* = \frac{B - 2Ab_1}{a}, C^* = \frac{C - 2Ab_2}{a}$$

and

$$D^* = D + A \left(\frac{b_1^2 + b_2^2}{a^2} \right) - \frac{Bb_1}{a} - \frac{Cb_2}{a}$$

Thus circles are invariant under translation, rotation and stretching.

Que 4.30. Discuss in brief about inversion and reflection transformation.

Answer

$$\text{Consider, } w = \frac{1}{z} \text{ for } z \neq 0 \quad \dots(4.30.1)$$

In polar coordinates,

$$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

So $R = \frac{1}{r}$, $\phi = -\theta$. Thus this transformation consists of an inversion in the unit circle ($Rr = 1$) followed by a mirror reflection about the real axis.

Also $|w| = \frac{1}{|z|}$. So the unit circle $|z| = 1$ maps onto the unit circle

$|w| = \frac{1}{1} = 1$. Further the interior of the unit circle $|z| = 1$ (point lying within $|z| = 1$) are transformed to the exterior of the unit circle $|w| = 1$ (points lying outside $|w| = 1$) or vice versa (Fig. 4.30.1).



Fig. 4.30.1.

Que 4.31. Find and plot the image of triangular region with vertices at $(0, 0)$, $(1, 0)$, $(0, 1)$ under the transformation $w = (1 - i)z + 3$ (Fig. 4.31.1).



Fig. 4.31.1.

Answer

Here,

$$u + iv = w = (1 - i)(x + iy) + 3$$

$$= x + iy - ix + y + 3$$

So

$$u(x, y) = x + y + 3, \quad v(x, y) = y - x$$

or

$$u = -v + 3$$

$\therefore$

$$v = 3 - u \text{ gives } A^*B^*$$

or

$$\text{At } AC, x = 0, u = y + 3, v = y,$$

$\therefore$

$$u = v + 3$$

$$v = u - 3 \text{ gives } A^*C^*$$

At BC , $x + y = 1$, or substituting $u = (x + y) + 3$

$$= 1 + 3 = 4,$$

$$u = 4 \text{ gives } B^*C^*$$

So the image is the triangular region with vertices at $A^*(3, 0)$, $B^*(4, -1)$,

$C^*(4, 1)$. Let $D\left(\frac{1}{4}, \frac{1}{4}\right)$ be any interior point of ABC . Its image is $D^*(3.5, 0)$ which is also an interior point of $A^*B^*C^*$.

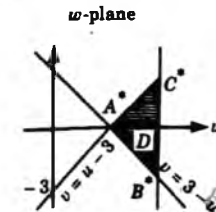


Fig. 4.31.2.

Que 4.32. Find the graph for the strip $1 < x < 2$ under the mapping

$$w = \frac{1}{z} \text{ (Fig. 4.32.1).}$$

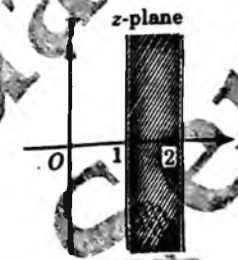


Fig. 4.32.1.

Answer

Here,

$$u + iv = w = \frac{1}{z} = \frac{x - iy}{x^2 + y^2}$$

So

$$x = \frac{u}{u^2 + v^2}, \quad y = \frac{-v}{u^2 + v^2}$$

Since $1 < x < 2$ so $1 < \frac{u}{u^2 + v^2} < 2$

$$\text{or } u^2 + v^2 - u < 0 \text{ and } 2(u^2 + v^2) - u > 0$$

Rewriting $\left(u - \frac{1}{2}\right)^2 + v^2 < \frac{1}{4}$ and $\left(u - \frac{1}{4}\right)^2 + v^2 > \frac{1}{16}$

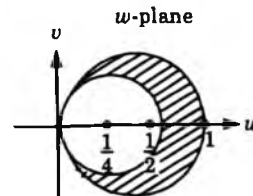


Fig. 4.32.2.

or $\left|w - \frac{1}{2}\right| < \frac{1}{2}$ and $\left|w - \frac{1}{4}\right| > \frac{1}{4}$

i.e., interior of the circle with centre at $\left(\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$ and exterior

of the circle with centre at $\left(\frac{1}{4}, 0\right)$ and radius $\frac{1}{4}$.

Thus the infinite strip maps to the region shaded in the w -plane (Fig. 4.32.2).

Que 4.33. Prove that $w = \frac{z}{1-z}$ maps the upper half of the z -plane onto upper half of the w -plane. What is the image of the circle $|z| = 1$ under this transformation? **AKTU 2021-22, Marks 10**

Answer

Given transformation

$$w = \frac{z}{1-z} \quad \dots(4.33.1)$$

i. Given region is the upper half of the z -plane $y > 0$...(4.33.2)

ii. Given region in the z -plane is the unit circle $|z| = 1$...(4.33.3)

To express x, y in terms of u, v :

Consider the transformation given in eq. (4.33.1) we get

$$w = \frac{z}{1-z}$$

Rewrite as

$$z = \frac{w}{1+w}$$

$$x + iy = \frac{u + iv}{1 + u + iv} \quad [\text{Using } w = u + iv \text{ and } z = x + iy]$$

$$x + iy = \frac{(u + iv)(1 + u - iv)}{(1 + u + iv)(1 + u - iv)}$$

[Multiplying and dividing by the conjugate]

$$x + iy = \frac{(u + u^2 + v^2 + iv)}{(1 + u)^2 + v^2}$$

Equating the real part and imaginary part on both sides, we get

$$x = \frac{(u + u^2 + v^2)}{(1 + u)^2 + v^2} \text{ and } y = \frac{v}{(1 + u)^2 + v^2} \quad \dots(4.33.4)$$

To find the image of the given region in z -plane:

Region (i) $y > 0$

Consider the given region in eq. (4.33.2), we get

$$y > 0$$

$$\frac{v}{(1 + u)^2 + v^2} > 0 \quad [\text{Using eq. (4.33.4)}]$$

$$v > 0$$

Hence, the transformation $w = \frac{z}{1-z}$ maps the upper half of the

z -plane into the upper half of the w -plane.

Region (ii) $|z| = 1$

Consider the given region in eq. (4.33.3), we get

$$\begin{aligned} |z| &= 1 \\ x^2 + y^2 &= 1 \end{aligned}$$

$$\frac{(u + u^2 + v^2)^2}{((1 + u)^2 + v^2)^2} + \frac{v^2}{((1 + u)^2 + v^2)^2} = 1 \quad [\text{Using eq. (4.33.4)}]$$

$$\frac{(u^2 + u^4 + v^4 + 2u^3 + 2u^2v^2 + 2uv^3 + v^4)}{(1 + u^2 + 2u + v^2)^2} = 1$$

$$\frac{(u^2(1 + u^2 + 2u + v^2) + v^2(1 + u^2 + 2u + v^2))}{(1 + u^2 + 2u + v^2)^2} = 1$$

$$\frac{(u^2 + v^2)}{(1 + u^2 + 2u + v^2)} = 1$$

$$u^2 + v^2 = 1 + u^2 + 2u + v^2$$

$$1 + 2u = 0$$

Hence, the image of the circle $|z| = 1$ under the given transformation is a straight line $1 + 2u = 0$ in w -plane.

PART-5

Mobius Transformation and their Properties.

CONCEPT OUTLINE

Mobius Transformation: It is also known as bilinear transformation. Bilinear transformation is the function w of a complex variable z of the form

$$w = f(z) = \frac{az + b}{cz + d}$$

Where a, b, c, d are complex or real constants subject to $ad - bc \neq 0$.

Properties of Mobius or Bilinear Transformation:

1. Circles are transformed into circles under bilinear transformation.
2. The cross-ratio of four points is invariant under a bilinear transformation.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.34. How could you determine the bilinear transformation?

Answer

1. A bilinear transformation can be uniquely determined by three given conditions.
2. To find the unique bilinear transformation which maps three given distinct points z_1, z_2, z_3 onto three distinct images w_1, w_2, w_3 , consider w which is the image of a general point z under this transformation.
3. We know that the cross-ratio of four points is invariant under a bilinear transformation, the cross-ratio of the four point w_1, w_2, w_3, w must be equal to the cross-ratio of z_1, z_2, z_3, z . Hence the unique bilinear transformation that maps three given point z_1, z_2, z_3 on to three given images w_1, w_2, w_3 is given by,

$$\frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)}$$

Que 4.35. Find Mobius transformation that maps points

$z = 0, -i, 2i$ into the points $w = 5i, \infty, -\frac{i}{3}$ respectively.

AKTU 2018-19, Marks 10

Answer

We know that,

$$\frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)} = \frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)}$$

We have $w_1 = 5i, w_2 = 0, w_3 = -i/3$
and $z_1 = 0, z_2 = -i, z_3 = 2i$

By putting these values,

$$\frac{(0 + i)(2i - z)}{(0 - z)(2i + i)} = \frac{(5i - w_2)\left(-\frac{i}{3} - w\right)}{(5i - w)\left(-\frac{i}{3} - w_1\right)}$$

If one of points, in this case $w_2 = \infty$, then the quotient of the two differences

which contain w_2 i.e., $\left(\frac{5i - w_2}{-\frac{i}{3} - w_1}\right)$ is replaced by 1.

$$\frac{z - 2i}{3z} = \lim_{w_2 \rightarrow \infty} \frac{\left(\frac{5i}{w_2} - 1\right)\left(\frac{i}{3} + w\right)}{(5i - w)\left(-\frac{i}{3w_2} + 1\right)} \quad (\because w_2 = \infty)$$

$$\frac{z - 2i}{3z} = \frac{\frac{i}{3} + w}{w - 5i}$$

Solving for w , we get

$$\frac{2i}{z} = \frac{2w + 6i}{5i - w}$$

or

$$w = \frac{-3iz - 5}{z + i} = \frac{-3z + 5i}{-iz + 1}$$

Que 4.36. Find a bilinear transformation which maps the point $i, -i, 1$ of the z -plane into $0, 1, \infty$ of the w -plane respectively.

AKTU 2021-22, Marks 10

Answer

We know that,

$$\frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)}$$

We have $w_1 = 0, w_2 = 1, w_3 = \infty$

and $z_1 = i, z_2 = -i, z_3 = 1$

By putting these values,

$$\frac{(0 - 1)(w_3 - w)}{(0 - w)(w_3 - 1)} = \frac{(i + i)(1 - z)}{(i - z)(1 + i)}$$

$$\frac{(w_3 - w)}{w(w_3 - 1)} = \frac{2i(1 - z)}{(i - z)(1 + i)}$$

The quotient of the differences which contain w_3 i.e., $\left(\frac{w_3 - w}{w_3 - 1}\right)$ is replaced by 1.

$$\frac{2i(1 - z)}{(i - z)(1 + i)} = \lim_{w_3 \rightarrow \infty} \left(\frac{1 - \frac{w}{w_3}}{1 - \frac{1}{w_3}} \right) w$$

Since $w_3 \rightarrow \infty$, $\frac{1}{w_3}$ and $\frac{w}{w_3}$ became 0

So,
$$\frac{2i(1 - z)}{(i - z)(1 + i)} = \frac{1}{w}$$

$$w = \frac{(i - z)(1 + i)}{2i(1 - z)}$$



Complex Variable Integration

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PART-1**Complex Integration.****Questions-Answers****Long Answer Type and Medium Answer Type Questions****Que 5.1.** How to evaluate complex integration?**Answer**

We shall consider the line integral of a complex function. Consider a continuous function $f(z)$ of the complex variable $z = x + iy$ defined at all points of a curve C having end points A and B . Divide C into n parts at the points

$$A = P_0(z_0), P_1(z_1), \dots, P_i(z_i), \dots, P_n(z_n) = B$$

**Fig. 5.1.1.**

Let $\delta z_i = z_i - z_{i-1}$ and ζ_i be any point on the arc $P_{i-1}P_i$. The limit of the sum $\sum_{i=1}^n f(\zeta_i) \delta z_i$ as $n \rightarrow \infty$ in such a way that the length of the chord δz_i approaches zero, is called the line integral of $f(z)$ taken along the path C , i.e.,

$$\int_C f(z) dz$$

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that $dz = dx + idy$,

$$\int_C f(z) dz = \int_C (u dx - v dy) + i \int_C (v dx + u dy)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

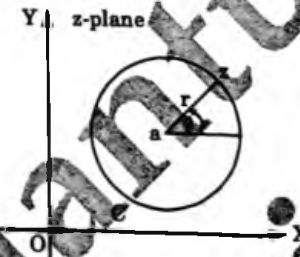
Qus 5.2. Prove that

$$\text{i. } \int_C \frac{dz}{z-a} = 2\pi i. \quad \text{ii. } \int_C (z-a)^n dz = 0 \quad [n, \text{ any integer } \neq -1]$$

where, C is the circle $|z-a| = r$.

Answer

The parametric equation of C is $z-a = re^{i\theta}$, where θ varies from 0 to 2π as z describes C once in the positive (anti-clockwise) sense.

**Fig. 5.2.1.**

$$\text{i. } \int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{e^{i\theta}} i r e^{i\theta} d\theta \quad [\because dz = i r e^{i\theta} d\theta]$$

$$= i \int_0^{2\pi} d\theta = 2\pi i$$

$$\text{ii. } \int_C (z-a)^n dz = \int_0^{2\pi} r^n e^{in\theta} i r e^{i\theta} d\theta$$

$$= i r^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta = \frac{r^{n+1}}{n+1} [e^{i(n+1)\theta}]_0^{2\pi}$$

provided $n \neq -1$

$$= \frac{r^{n+1}}{n+1} [e^{2\pi i(n+1)} - 1] = 0 \quad [\because e^{2\pi i(n+1)} = 1]$$

Que 5.3. Evaluate $\int_0^{1+i} (\bar{z})^2 dz$, along

i. The line $y = x/2$,

ii. The real axis to 2 and then vertically to $2 + i$.

Answer

- i. Along the line OA , $x = 2y$, $z = (2 + i)y$, $\bar{z} = (2 - i)y$ and $dz = (2 + i) dy$ Fig. 5.3.1.

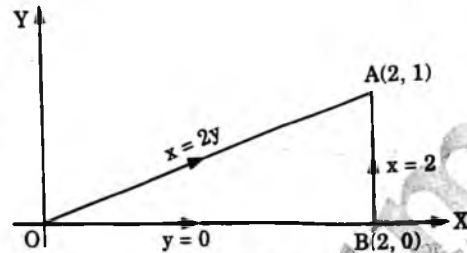


Fig. 5.3.1.

$$I = \int_0^{2+i} (\bar{z})^2 dz = \int_0^1 (2-i)^2 y^2 (2+i) dy$$

$$= 5(2-i) \left[\frac{y^3}{3} \right]_0^1 = \frac{5}{3}(2-i)$$

- ii. Now along OB , $z = x$, $\bar{z} = x$, $dz = dx$;
and along BA , $z = 2 + iy$, $\bar{z} = 2 - iy$, $dz = i dy$

$$\begin{aligned} I &= \int_0^2 x^2 dx + \int_0^1 (2-iy)^2 i dy \\ &= \left[\frac{x^3}{3} \right]_0^2 + \int_0^1 [4y + (4-y^2)i] dy \\ &= \frac{8}{3} + 4 \cdot \frac{1}{2} + \left(4 - \frac{1}{3} \right) i = \frac{1}{3}(14 + 11i). \end{aligned}$$

PART-2

Cauchy-Integral Theorem, Cauchy Integral Formula.

CONCEPT OUTLINE

Cauchy's Integral Theorem : If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within a simple closed curve C , then

$$\oint_C f(z) dz = 0$$

For multiple connected regions,

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz$$

when integral along each curve is taken in anticlockwise direction.
Cauchy's Integral Formula : If $f(z)$ is analytic within and on a closed curve C and if a is any point within C , then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)} dz$$

Cauchy's Integral Formula for Derivative of an Analytic Function : If a function $f(z)$ is analytic in a domain D , then at any point $z = a$ of D , $f(z)$ has derivatives of all orders, all of which are again analytic functions in D and are given by

$$f'(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

Where, C is any closed curve in D surrounding the point $z = a$.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.4. State Cauchy's Integral theorem and derive it.

Answer

- A. **Statement :** If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a simple closed curve C , then

$$\oint_C f(z) dz = 0$$

- B. **Proof :** Let R be the region bounded by the curve C .



Fig. 5.4.1.

Let,

$$f(z) = u(x, y) + iv(x, y), \text{ then}$$

$$\begin{aligned} \oint_C f(z) dz &= \oint_C (u + iv)(dx + i dy) \\ &= \oint_C (u dx - v dy) + i \oint_C (v dx + u dy) \quad \dots(5.4.1) \end{aligned}$$

Since $f'(z)$ is continuous, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in R . Hence by Green's theorem, we have

$$\oint_C f(z) dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy \quad \dots(5.4.2)$$

Now $f(z)$ being analytic at each point of the region R , by Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, the two double integrals in eq. (5.4.2) vanish.

$$\text{Hence} \quad \oint_C f(z) dz = 0$$

Que 5.5. State and prove Cauchy's integral formula.

AKTU 2017-18 (III), Marks 10

Answer

- A. Statement :** If $f(z)$ is analytic within and on a closed curve C and a is any point within C , then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

- B. Proof :** Consider the function $\frac{f(z)}{z-a}$, which is analytic at every point within C except at $z=a$. Draw a circle C_1 with a as centre and radius ρ such that C_1 lies entirely inside C . Thus $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1 .

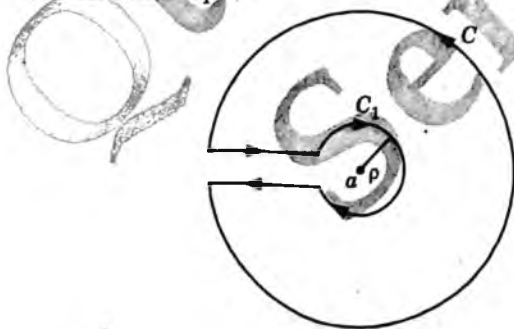


Fig. 5.5.1.

By Cauchy's theorem, we have

$$\oint_C \frac{f(z)}{z-a} dz = \oint_{C_1} \frac{f(z)}{z-a} dz \quad \dots(5.5.1)$$

Now, the equation of circle C_1 is $|z-a| = \rho$ or $z-a = \rho e^{i\theta}$
So that $dz = i\rho e^{i\theta} d\theta$

$$\therefore \oint_{C_1} \frac{f(z)}{z-a} dz = \int_0^{2\pi} \frac{f(a + \rho e^{i\theta})}{\rho e^{i\theta}} i \rho e^{i\theta} d\theta = i \int_0^{2\pi} f(a + \rho e^{i\theta}) d\theta$$

Hence by eq. (5.5.1), we have

$$\oint_C \frac{f(z)}{z-a} dz = i \int_0^{2\pi} f(a + \rho e^{i\theta}) d\theta \quad \dots(5.5.2)$$

In the limiting form, as the circle C_1 shrinks to the point a , i.e., $\rho \rightarrow 0$, then from eq. (5.5.2),

$$\oint_C \frac{f(z)}{z-a} dz = i \int_0^{2\pi} f(a) d\theta = i f(a) \int_0^{2\pi} d\theta = 2\pi i f(a)$$

$$\text{Hence} \quad f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

Que 5.6. State Cauchy integral theorem for an analytic function. Verify this theorem by integrating the function $z^3 + iz$ along the boundary of the rectangle with vertices $1, -1, i, -i$.

AKTU 2014-15 (III), Marks 05

Answer

- A. Cauchy's Integral Theorem :** Refer Q. 5.4, Page 5-5F, Unit-5.
B. Numerical :

$$\oint_C f(z) dz = \int_{AB} f(z) dz + \int_{BC} f(z) dz + \int_{CD} f(z) dz + \int_{DA} f(z) dz$$

$$\int_{AB} f(z) dz = \int_{-1}^1 (x+iy)^3 + i(x+iy)(dx+idy) = 0 \quad \dots(5.6.1)$$

$$\int_{BC} f(z) dz = \int_0^1 ((x+(x-1))^3 + i(2x-1)(2dx))$$

$$= 2 \int_0^1 ((2x-1)^3 + i(2x-1)) dx = -i \quad \dots(5.6.2)$$

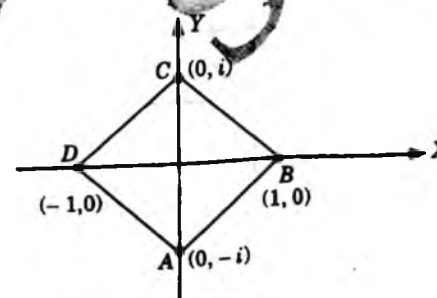


Fig. 5.6.1.

$$\int_{CD} f(z) dz = \int [(x+i(i x+i))^3 + i(-ix+i)](0) = 0 \quad \dots(5.6.3)$$

$$\begin{aligned} \int_{DA} f(z) dz &= 2 \int [(x-i(i x+i))^3 + i(2x+1)] dx \\ &= 2 \int [(2x+1)^3 + i(2x+1)] dx \\ &= 2 \left[\frac{(2x+1)^4}{8} + i \frac{(2x+1)^2}{2} \right]_1^0 = i \quad \dots(5.6.4) \end{aligned}$$

From eq. (5.6.1), eq. (5.6.2), eq. (5.6.3) and eq. (5.6.4), we have

$$\int_C f(z) dz = -i + 0 + 0 + i = 0 \text{ (Hence proved)}$$

Que 5.7. Verify Cauchy's theorem by integrating e^z along the boundary of the triangle with the vertices at the points $1+i$, $-1+i$ and $-1-i$. **AKTU 2017-18 (II), Marks 10**

Answer

$$\oint_C f(z) dz = \int_{AB} f(z) dz + \int_{BC} f(z) dz + \int_{CA} f(z) dz \quad \dots(5.7.1)$$

Along AB, $y = x, dy = dx$
 $f(z) = e^{iz} = e^{i(x+iy)}$
 $f(x) = e^{i(1+ix)}$

$$\begin{aligned} \int_{AB} f(z) dz &= \int_{-1}^1 e^{i(1+ix)} (dx + i dx) \\ &= (1+i) \left[\frac{e^{i(1+ix)}}{i(1+i)} \right]_{-1}^1 = \frac{(1+i)}{(i-1)} [e^{i-1} - e^{-i+1}] \end{aligned}$$

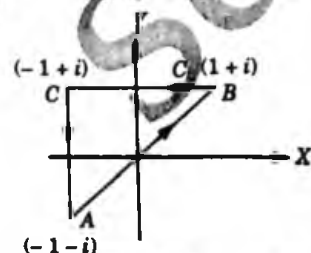


Fig. 5.7.1.

Along BC, $y = 1, dy = 0$

$$\int_{BC} f(z) dz = \int_1^{-1} e^{i(x+i)} dx = e^{-1} \int_1^{-1} e^{ix} dx = \frac{1}{ie} (e^{-i} - e^i)$$

Along CA, $x = -1, dx = 0$

$$\begin{aligned} \int_{CA} f(z) dz &= \int_1^{-1} e^{i(-1+iy)} i dy = ie^{-i} \int_1^{-1} e^{-y} dy \\ &= -ie^{-i} (e^{+1} - e^{-1}) = -ie^{-i} (e - e^{-1}) \end{aligned}$$

From eq. (5.7.1)

$$\begin{aligned} \oint_C f(z) dz &= \frac{(i+1)^2}{-2} \left[\frac{e^i}{e} - ee^{-i} \right] + \frac{e^{-i}}{ie} - \frac{e^i}{ie} - ie^{-i}e + \frac{ie^{-i}}{e} \\ &= -\frac{ie^i}{e} + iee^{-i} + \frac{e^{-i}}{ie} - \frac{e^i}{ie} - ie^{-i}e + \frac{ie^{-i}}{e} \\ &= -ie^{i-1} + ie^{-i+1} - ie^{i-1} + ie^{i-1} - ie^{-i+1} + ie^{-i-1} \\ \therefore \oint_C f(z) dz &= 0 \text{ (Hence proved)} \end{aligned}$$

Que 5.11 State Cauchy's integral formula. Hence,

Evaluate $\oint_C \frac{dz}{z^3(z^3-4)e^z}$, where C is $|z|=1$

AKTU 2012-13 (IV), Marks 05

Answer

- A. Cauchy's Integral Formula : Refer Q. 5.5, Page 5-6F, Unit-5.
 B. Numerical:

Let, $I = \oint_C \frac{dz}{z^3(z^3-4)e^z}, C = |z|=1$

$$I = \oint_C \frac{e^{-z} dz}{z^3(z^3-4)}$$

Poles are $z = 0$ (of order 2), $z = \pm 2$
 $z = 0$ is the only pole which lie inside C .

$$\therefore I = \oint_C \frac{e^{-z} / (z^3-4)}{z^3} dz = 2\pi i \left[\frac{d}{dz} \left(\frac{e^{-z}}{z^3-4} \right) \right]_{z=0}$$

$$I = 2\pi i \left[\frac{-(z^3-4)e^{-z} - 2ze^{-z}}{(z^3-4)^2} \right]_{z=0}$$

$$I = -2\pi i \left[\frac{-4+0}{16} \right]$$

$$I = \frac{\pi i}{2}$$

Thus, $\oint_C \frac{1}{z^2(z^2-4)e^z} dz = \frac{\pi i}{2}$

Que 5.9. State Cauchy's integral formula. Hence evaluate :

$$\int_C \frac{2z+1}{z^2+z} dz, \text{ where } C \text{ is } |z| = \frac{1}{2}.$$

AKTU 2014-15 (IV), Marks 05

Answer

A. Cauchy Integral Formula : Refer Q. 5.5, Page 5-6F, Unit-5.

B. Numerical :

Let, $I = \int_C \frac{2z+1}{z^2+z} dz$

Poles are given by, $z^2+z=0$

$$z(z+1)=0$$

$$z=0, z=-1$$

$|z| = \frac{1}{2}$ is a circle with centre at origin and radius $\frac{1}{2}$. Pole $z=0$ enclosed in $|z| = \frac{1}{2}$.

$$\int_C \frac{2z+1}{z(z+1)} dz = \int_C \frac{z+1}{z} = 2\pi i \left[\frac{2z+1}{z+1} \right]_{z=0}$$

$$\int_C \frac{2z+1}{z(z+1)} dz = 2\pi i$$

Que 5.10. Use Cauchy's integral formula to show that

$$\int_C \frac{e^{it}}{z^2+1} dz = 2\pi i \sin t \text{ if } t > 0 \text{ and } C \text{ is the circle } |z| = 3.$$

AKTU 2013-14 (IV), Marks 05

Answer

Poles of the integrand are given by

$$z^2+1=0, z=\pm i \text{ (order 1)}$$

The circle $|z| = 3$ has centre at $z=0$ and radius 3. It encloses both the singularities $z=i$ and $z=-i$.

Now, $\int_C \frac{e^{it}}{z^2+1} dz = \int_C \frac{e^{it}}{(z+i)(z-i)} dz$

$$= \int_{C_1} \frac{\left(\frac{e^{it}}{z-i}\right)}{z+i} dz + \int_{C_2} \frac{\left(\frac{e^{it}}{z+i}\right)}{z-i} dz = 2\pi i \left(\frac{e^{it}}{z+i}\right) \Big|_{z=-i} + 2\pi i \left(\frac{e^{it}}{z-i}\right) \Big|_{z=i}$$

$$= \pi (e^{it} - e^{-it}) = 2\pi i \sin t$$

Que 5.11. Evaluate by Cauchy's integral formula

$$\oint_C \frac{z^2-2z}{(z+1)^2(z^2+4)} dz, \text{ where } C \text{ is the circle } |z| = 3.$$

AKTU 2015-16 (III), Marks 05

Answer

Here, we have $\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)} dz$

The poles are determined by putting the denominator equal to zero.

$$(z+1)^2(z^2+4)=0$$

$$z = -1, -1 \text{ and } z = \pm 2i$$

The circle $|z| = 3$ with centre at origin and radius = 3 encloses a pole at $z = -1$ of second order and simple poles $z = \pm 2i$.

Let the given integral $= I_1 + I_2 + I_3$ (5.11.1)

$$I_1 = \int_{C_1} \frac{z^2-2z}{(z+1)^2(z^2+4)} dz = \int_{C_1} \frac{z^2-2z}{(z+1)^2} dz$$

$$= 2\pi i \left[\frac{d}{dz} \frac{z^2-2z}{z^2+4} \right]_{z=-1}$$

$$= 2\pi i \left[\frac{(z^2+4)(2z-2) - (z^2-2z)2z}{(z^2+4)^2} \right]_{z=-1}$$

$$= 2\pi i \left[\frac{(1+4)(-2-2) - (1+2)2(-1)}{(1+4)^2} \right]$$

$$= 2\pi i \left(-\frac{14}{25} \right) = -\frac{28\pi i}{25}$$

$$I_2 = \int_{C_2} \frac{z^2-2z}{(z+1)^2(z+2i)} dz = 2\pi i \left[\frac{z^2-2z}{(z+1)^2(z+2i)} \right]_{z=2i}$$

$$= 2\pi i \left[\frac{-4-4i}{(2i+1)^2(2i+2i)} \right] = 2\pi i \frac{(1+i)}{4+3i}$$

$$I_3 = \int_{C_3} \frac{z^2 - 2z}{(z+1)^2(z-2i)} dz = 2\pi i \left[\frac{z^2 - 2z}{(z+1)^2(z-2i)} \right]_{z=-2i}$$

$$= 2\pi i \left[\frac{-4 + 4i}{(-2i+1)^2(-2i-2i)} \right] = 2\pi i \frac{(i-1)}{(3i-4)}$$

Now putting the value of I_1 , I_2 and I_3 in eq. (5.11.1), we get

$$\int_C \frac{z^2 - 2z}{(z+1)^2(z^2+4)} dz = \frac{-28\pi i}{25} + 2\pi i \left(\frac{1+i}{4+3i} \right) + 2\pi i \left(\frac{i-1}{3i-4} \right)$$

$$= 2\pi i \left[\frac{-14}{25} + \frac{1+i}{(4+3i)} + \frac{(i-1)}{(3i-4)} \right]$$

$$= 2\pi i \left[\frac{-14}{25} + \frac{(1+i)(3i-4) + (i-1)(4+3i)}{(-9-16)} \right]$$

$$= \frac{2\pi i}{-25} [14 + (3i-4-3-4i) + (4i-3-4-3i)]$$

$$= 0$$

Que 5.12. Evaluate the integral $\int \frac{e^{2z}}{(z+1)^5} dz$, around the boundary of the circle $|z| = 2$. **AKTU 2014-15 (III), Marks 05**

Answer

The given integral is $\int \frac{e^{2z}}{(z+1)^5}$

Poles are $z = -1$ of order 5 will lie in $|z| = 2$

Using Cauchy integral formula, we get

$$\int \frac{e^{2z}}{(z+1)^5} dz = \frac{2\pi i}{4!} \left[\frac{d^4}{dz^4} (e^{2z}) \right]_{z=-1}$$

$$= \frac{2\pi i}{4!} (16e^{2z})_{z=-1} = \frac{32\pi i}{24} \times e^{-2} = \frac{4\pi i}{3e^2}$$

Que 5.13. Using Cauchy integral formula evaluate

$$\int \frac{\sin z}{(z^2 + 25)^2} dz \text{ when } z \text{ is circle } |z| = 8.$$

AKTU 2018-19, Marks 10

Answer

Let, $I = \oint_C \frac{\sin z}{(z^2 + 25)^2} dz$

Given integral has poles $z = \pm 5i$ order 2.

$$\text{Now, } \oint_C \frac{\sin z}{(z^2 + 5^2)^2} dz = \oint_{C_1} \frac{\sin z}{(z^2 - 5^2)^2} dz + \oint_{C_2} \frac{\sin z}{(z^2 + 5^2)^2} dz$$

$$= 2\pi i \left[\frac{d}{dz} \left\{ \frac{\sin z}{(z+5i)^2} \right\} \right]_{z=5i} + 2\pi i \left[\frac{d}{dz} \left\{ \frac{\sin z}{(z-5i)^2} \right\} \right]_{z=-5i}$$

$$= 2\pi i \left[\frac{\cos z (z+5i)^2 - 2 \sin z (z+5i)}{(z+5i)^4} \right]_{z=5i}$$

$$+ 2\pi i \left[\frac{\cos z (z-5i)^2 - 2 \sin z (z-5i)}{(z-5i)^4} \right]_{z=-5i}$$

$$= 2\pi i \left[\frac{\cos 5i (10i)^2 - 2 \sin 5i (10i)}{(10i)^4} \right]$$

$$+ 2\pi i \left[\frac{\cos 5i (-10i)^2 - 2 \sin 5i (-10i)}{(-10i)^4} \right]$$

$$= \frac{2\pi i}{(10i)^4} [-100 \cos 5i - 20i \sin 5i - (-100 \cos 5i + 20i \sin 5i)]$$

$$= \frac{2\pi i}{10000} [-100 \cos 5i - 20i \sin 5i + 100 \cos 5i - 20i \sin 5i]$$

$$= \frac{2\pi i}{10000} (-40i \sin 5i) = \frac{\pi}{125} \sin 5i$$

Que 5.14. Evaluate $\int_C \frac{(1+z)\sin z}{(2z-3)^2} dz$, where C is the circle

$|z - i| = 2$ counter clockwise.

AKTU 2013-14 (III), Marks 05

Answer

The given integral is $\int \frac{(1+z)\sin z}{(2z-3)^2} dz$

Poles of integrand,

$$(2z-3)^2 = 0$$

$$z = \frac{3}{2}, \frac{3}{2}$$

$\therefore$ Pole lie inside the circle of radius 2.

By Cauchy's integral formula,

$$\begin{aligned}\int_C \frac{(1+z)\sin z}{(2z-3)^2} dz &= 2\pi i \left[\frac{d}{dz} (1+z)\sin z \right]_{z=3/2} \\ &= 2\pi i [(1+z)\cos z + \sin z]_{z=3/2} \\ &= 2\pi i \left(\frac{5}{2} \cos \frac{3}{2} + \sin \frac{3}{2} \right)\end{aligned}$$

Que 5.15. Evaluate $\int_0^{3+i} (\bar{z})^2 dz$, along the real axis from $z = 0$ to $z = 3$ and then along a line parallel to imaginary axis from $z = 3$ to $z = 3 + i$. **AKTU 2012-13 (IV), Marks 05**

Answer

$$\int_0^{3+i} (\bar{z})^2 dz = \int_0^{3+i} (x-iy)^2 (dx+idy) = \int_{OA} x^2 dx + \int_{AB} (3-iy)^2 idy$$



Fig. 5.15.1

Along OA, $y = 0$, $dy = 0$, x varies 0 to 3

Along AB, $x = 3$, $dx = 0$, and y varies 0 to 1

$$\begin{aligned}\therefore \int_0^{3+i} (\bar{z})^2 dz &= \int_0^3 x^2 dx + \int_0^1 (3-iy)^2 idy = \left[\frac{x^3}{3} \right]_0^3 + i \int_0^1 (9 - 6iy - y^2) dy \\ &= \left[\frac{x^3}{3} \right]_0^3 + i \left[9y - 3y^2 - \frac{y^3}{3} \right]_0^1 = \frac{27}{3} + i \left[\frac{26}{3} - 3i \right] = 12 + \frac{26i}{3}\end{aligned}$$

Que 5.16. Integrate $f(z) = \operatorname{Re}(z)$ from $z = 0$ to $z = 1 + 2i$, (i) along straight line joining $z = 0$ to $z = 1 + 2i$, (ii) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + 2i$. **AKTU 2013-14 (III), Marks 06**

Answer

i. $\int_0^{1+2i} f(z) dz = \int_0^{1+2i} \operatorname{Re}(z) dz$

Equation of OB is,

$$y - 0 = \frac{2-0}{1-0}(x-0)$$

$$y = 2x$$

$$dy = 2dx$$

$$z = x + iy$$

$$dz = dx + idy = dx + i(2dx)$$

$$\int_0^{1+2i} \operatorname{Re}(z) dz = \int_0^{1+2i} x(dx+idy)$$

$$= \int_0^1 x(dx+2idy) = (1+2i) \int_0^1 x dx = (1+2i) \left[\frac{x^2}{2} \right]_0^1 = \frac{1+2i}{2}$$



Fig. 5.16.1

ii. $\int f(z) dz = \int_{OA} \operatorname{Re}(z) dz + \int_{AB} \operatorname{Im}(z) dz$

$$= \int_0^1 x dx + \int_0^2 1(idy) = \left[\frac{x^2}{2} \right]_0^1 + i \left[y \right]_0^2 = \frac{1}{2} + 2i = \frac{1+4i}{2}$$



Fig. 5.16.2

Que 5.17. Evaluate: $\int_0^{\infty} \frac{\sin mx}{x} dx$, $m > 0$.

AKTU 2017-18 (IV), Marks 10

Answer

Since $\sin mx$ is the imaginary part of e^{imx} , we consider the function

$$\phi(z) = \frac{e^{imz}}{z}$$

This has a simple pole at $z = 0$, which lies on the real axis. Enclose this singularity by a small semi-circle C_2 : $|z| = r$. Since the function has no singularity within this contour, by Cauchy's theorem, we have

$$\oint_C \phi(z) dz = 0$$

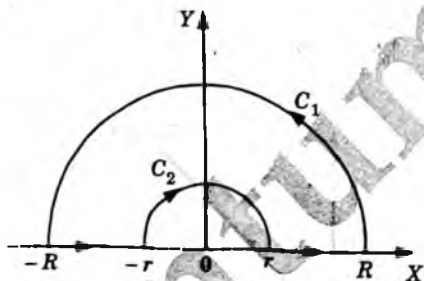


Fig. 5.17.1.

$$\int_{-R}^{-r} \phi(x) dx + \int_{C_2} \phi(z) dz + \int_r^R \phi(x) dx + \int_{C_1} \phi(z) dz = 0$$

$$\int_{-R}^{-r} \frac{e^{imx}}{x} dx + \int_{C_2} \frac{e^{imz}}{z} dz + \int_r^R \frac{e^{imx}}{x} dx + \int_{C_1} \frac{e^{imz}}{z} dz = 0 \quad \dots(5.17.1)$$

Substituting $-x$ for x in the first integral and combining it with the third integral, we get

$$\int_r^R \frac{e^{imx} - e^{-imx}}{x} dx + \int_{C_2} \frac{e^{imz}}{z} dz + \int_{C_1} \frac{e^{imz}}{z} dz = 0$$

or $2i \int_r^R \frac{\sin mx}{x} dx + \int_{C_2} \frac{e^{imz}}{z} dz + \int_{C_1} \frac{e^{imz}}{z} dz = 0 \quad \dots(5.17.2)$

$$\text{Now } \int_{C_1} \frac{e^{imz}}{z} dz = \int_{C_1} \frac{1}{z} dz + \int_{C_1} \frac{e^{imz} - 1}{z} dz \quad \dots(5.17.3)$$

On C_2 , $z = re^{i\theta}$

$$\therefore \int_{C_2} \frac{1}{z} dz = \int_{\pi}^0 \frac{re^{i\theta} i d\theta}{re^{i\theta}} = - \int_0^{\pi} i d\theta = -i\pi$$

$$\text{Also } \left| \int_{C_1} \frac{e^{imz} - 1}{z} dz \right| \leq M \int_{C_1} \left| \frac{dz}{z} \right| = \pi M$$

Where M is the maximum value on C_2 of $|e^{imz} - 1| = |e^{imr(\cos \theta + i \sin \theta)} - 1|$

Clearly, $M \rightarrow 0$ as $r \rightarrow 0$

$$\therefore \text{ From eq. (5.17.3), } \int_{C_1} \frac{e^{imz}}{z} dz = -i\pi$$

Putting $z = Re^{i\theta}$ in the integral over C_1 , we get

$$\int_{C_1} \frac{e^{imz}}{z} dz = \int_0^{\pi} \frac{e^{imR(\cos \theta + i \sin \theta)}}{Re^{i\theta}} Re^{i\theta} i d\theta = i \int_0^{\pi} e^{imR \cos \theta} e^{-mR \sin \theta} d\theta$$

$$\text{Since, } |e^{imR \cos \theta}| \leq 1$$

$$\therefore \left| \int_{C_1} \frac{e^{imz}}{z} dz \right| \leq \int_0^{\pi} e^{-mR \sin \theta} d\theta = 2 \int_0^{\pi/2} e^{-mR \sin \theta} d\theta$$

Also, $\frac{\sin \theta}{\theta}$ continually decreases from 1 to $2/\pi$ as θ increases from 0 to $\pi/2$.

$$\therefore \text{ For } 0 \leq \theta \leq \frac{\pi}{2}, \frac{\sin \theta}{\theta} \geq \frac{2}{\pi} \quad \text{or} \quad \sin \theta \geq \frac{2\theta}{\pi}$$

$$\therefore \left| \int_{C_1} \frac{e^{imz}}{z} dz \right| \leq 2 \int_0^{\pi/2} e^{-2mR \theta/\pi} d\theta = \left[-\frac{\pi}{mR} e^{-2mR \theta/\pi} \right]_0^{\pi/2} = \frac{\pi}{mR} (1 - e^{-mR})$$

$$\text{As } R \rightarrow \infty, \frac{\pi}{mR} (1 - e^{-mR}) \rightarrow 0$$

$$\therefore \int_{C_1} \frac{e^{imz}}{z} dz = 0$$

Hence from eq. (5.17.2), on taking the limit as $r \rightarrow 0$ and $R \rightarrow \infty$, we get

$$2i \int_0^{\infty} \frac{\sin mx}{x} dx - i\pi = 0$$

$$\int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}$$

Que 5.18. Evaluate $\oint_C \frac{e^z}{z(1-z)^3} dz$, where C is

i. $|z| = \frac{1}{2}$

ii. $|z - 1| = \frac{1}{2}$

iii. $|z| = 2$

AKTU 2021-22, Marks 10

Answer

i. Since $a = 0$ lies inside C , therefore, take $f(z) = \frac{e^z}{(1-z)^3}$

This function is analytic at all points inside C .

$\therefore$ By Cauchy's integral formula $f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$, we get

$$\int_c \frac{e^z}{(1-z)^3} dz = 2\pi i f(0) = 2\pi i (1) = 2\pi i$$

- ii. Since 1 lies inside c , take $f(z) = \frac{e^z}{z}$
 $f(z)$ is analytic at all points inside c .
 $\therefore$ By Cauchy's integral formula,

$$f''(1) = \frac{2!}{2\pi i} \int_c \frac{f(z)}{(z-1)^3} dz = -\frac{1}{\pi i} \int_c \frac{f(z)}{(z-1)^3} dz$$

$$\text{Hence } \int_c \frac{e^z}{z(1-z)^3} dz = \int_c \frac{f(z)}{(z-1)^3} dz = -\pi i f''(1)$$

$$= -\pi i \left[\frac{e^z(z^3 - 2z^2 + 2z)}{z^4} \right]_{z=1}$$

$$= -\pi i [e(1 - 2 + 2)] = -\pi i e$$

- iii. Given both 0 and 1 lies inside c .

By partial fractions

$$\frac{1}{z(1-z)^3} = \frac{A}{z} + \frac{B}{1-z} + \frac{C}{(1-z)^2} + \frac{D}{(1-z)^3}$$

On solving, $A = 1, B = -1, C = 5, D = 1$

$$\frac{1}{z(1-z)^3} = \frac{1}{z} - \frac{1}{1-z} + \frac{5}{(1-z)^2} + \frac{1}{(1-z)^3}$$

$$= \frac{1}{z} + \frac{1}{z-1} + \frac{5}{(z-1)^2} + \frac{1}{(z-1)^3}$$

$$\text{Hence } \int_c \frac{e^z}{z(1-z)^3} dz$$

$$= \int_c \frac{e^z}{z} dz + \int_c \frac{e^z}{z-1} dz + 5 \int_c \frac{e^z}{(z-1)^2} dz - \int_c \frac{e^z}{(z-1)^3} dz$$

$$= 2\pi i f(0) + 2\pi i f(1) + 5(2\pi i) f'(1) - 2\pi i f''(1)$$

$$= 2\pi i [f(0) + f(1) + 5f'(1) - f''(1)], \text{ where } f(z) = e^z$$

$$= 2\pi i [1 + e + 5e - e] = 2\pi i (1 + 5e)$$

PART-3

Taylor's and Laurent's Series.

CONCEPT OUTLINE

Taylor's Series: A function $f(z)$ which is analytic at all points within a circle C with centre at a can be represented uniquely as a convergent power series known as Taylor's series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Where,

$$a_n = \frac{f^{(n)}(a)}{n!}$$

Laurent's Series: If $f(z)$ is analytic inside and on the boundary of the annular (ring shaped) region R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2 ($r_1 > r_2$) respectively having centre at a , then for all z in R

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

Where,

$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw$$

and

$$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.19. Expand $\frac{1}{z^3 - 3z + 2}$ in the region $1 < |z| < 2$.

AKTU 2014-15 (IV), Marks 05

Answer

$$f(z) = \frac{1}{z^3 - 3z + 2} = \frac{1}{(z-2)(z-1)} = -\frac{1}{2} \left(\frac{1}{z-2} \right) + \frac{1}{2} \left(\frac{1}{z-1} \right)$$

$$= -\frac{1}{2} \left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots \right] + \frac{1}{2} \left[1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right]$$

After rearranging, we get,

$$f(z) = \dots - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 \dots$$

Que 5.20. Obtain the Taylor's series expansion of $f(z) = \frac{1}{z^3 - 4z + 3}$ about the point $z = 4$. Find its region of convergence.

AKTU 2013-14 (IV), Marks 05

Answer

If the centre of the circle is at $z = 4$, then the distances of the singularities $z = 1$ and $z = 3$ from centre are 3 and 1.

Hence if a circle is drawn with centre at $z = 4$ and radius 1 then within circle $|z - 4| = 1$, the given function $f(z)$ is analytic hence it can be expanded in Taylor's series within the circle $|z - 4| = 1$ which is therefore the region of convergence.

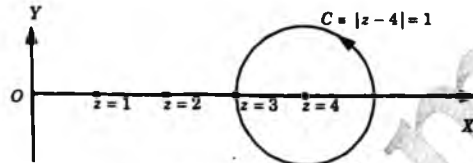


Fig. 5.20.1.

$$\begin{aligned} f(z) &= \frac{1}{(z-1)(z-3)} = \frac{1}{2} \left[\frac{1}{z-3} - \frac{1}{z-1} \right] = \frac{1}{2} \left[\frac{1}{z-4+1} - \frac{1}{z-4+3} \right] \\ &= \frac{1}{2} \left[(1+(z-4))^{-1} - \frac{1}{3} \left(1 + \left(\frac{z-4}{3} \right) \right)^{-1} \right] \\ f(z) &= \frac{1}{2} \left[\sum_{n=0}^{\infty} (-1)^n (z-4)^n - \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-4}{3} \right)^n \right] \end{aligned}$$

Que 5.21. Find the Taylor series expansion of the function $\tan^{-1} z$ about the point $z = \pi/4$.

AKTU 2014-15 (III), Marks 05

Answer

$$f(z) = \tan^{-1} z$$

$$f'(z) = \frac{1}{1+z^2}$$

$$f''(z) = \frac{-2z}{(1+z^2)^2}$$

$$f'''(z) = -2 \left[\frac{(1+z^2)^2 - 4z^2(1+z^2)}{(1+z^2)^4} \right] = -2 \left[\frac{1+z^2-4z^2}{(1+z^2)^3} \right] = \frac{2(3z^2-1)}{(1+z^2)^3}$$

$$f'\left(\frac{\pi}{4}\right) = \tan^{-1}\left(\frac{\pi}{4}\right) = 0.6658, f'\left(\frac{\pi}{4}\right) = 0.6185$$

$$f''\left(\frac{\pi}{4}\right) = \frac{-2(0.785)}{2.6142} = -0.60087$$

Thus,

$$\tan^{-1} z = 0.6658 + \left(z - \frac{\pi}{4}\right) (0.6185) + \frac{\left(z - \frac{\pi}{4}\right)^2}{2!} (-0.60087) + \dots$$

Que 5.22. Find all Taylor and Laurent series expansion of the following function about $z = 0$

$$f(z) = \frac{-2z+3}{z^2-3z+2}$$

AKTU 2013-14 (III), Marks 05

Answer

$$f(z) = \frac{-2z+3}{z^2-3z+2} = -\frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{(1-z)} + \frac{1}{2\left(1-\frac{z}{2}\right)} \quad \dots(5.22.1)$$

$$= (1-z)^{-1} + \frac{1}{2} \left(1-\frac{z}{2}\right)^{-1}$$

Now expanding by binomial expansion

$$f(z) = (1+z+z^2+z^3+\dots) + \frac{1}{2} \left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots \right]$$

$$\text{or } f(z) = \sum_{n=0}^{\infty} (1)^n z^n + \frac{1}{2} \sum_{n=0}^{\infty} (1)^n \left(\frac{z}{2}\right)^n$$

This is the Taylor's series expansion of given function.

Eq. (5.22.1) can also be written as,

$$f(z) = -\frac{1}{z} \left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z} \left(1-\frac{2}{z}\right)^{-1}$$

Now expanding by binomial expansion, we get

$$f(z) = -\frac{1}{z} \left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots \right] - \frac{1}{z} \left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \dots \right]$$

$$f(z) = -\frac{1}{z} \sum_{n=0}^{\infty} (1)^n \frac{1}{z^n} - \frac{1}{z} \sum_{n=0}^{\infty} (1)^n \left(\frac{2}{z}\right)^n$$

This is the Laurent's series expansion of given function.

Que 5.23. Find the Laurent series for the function

$$f(z) = \frac{7z^2 + 9z - 18}{z^3 - 9z}, z \text{ is complex variable valid for the regions}$$

- i. $0 < |z| < 3$ ii. $|z| > 3$

AKTU 2015-16 (IV), Marks 10

AKTU 2012-13 (IV), Marks 05

Answer

$$f(z) = \frac{7z^2 + 9z - 18}{z^3 - 9z}$$

Using partial fraction,

$$\frac{7z^2 + 9z - 18}{z^3 - 9z} = \frac{A}{z} + \frac{B}{z-3} + \frac{C}{z+3}$$

$$A = \frac{7z^2 + 9z - 18}{(z-3)(z+3)} \Big|_{z=0} = 2$$

$$B = \frac{7z^2 + 9z - 18}{z(z+3)} \Big|_{z=3} = 4$$

$$C = \frac{7z^2 + 9z - 18}{z(z-3)} \Big|_{z=-3} = 1$$

$$f(z) = \frac{7z^2 + 9z - 18}{z^3 - 9z} = \frac{2}{z} + \frac{4}{z-3} + \frac{1}{z+3}$$

- i. $0 < |z| < 3$

Rearrangement of function $f(z)$,

$$f(z) = \frac{2}{z} - \frac{4}{3} \left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{3} \left(1 + \frac{z}{3}\right)^{-1}$$

$$f(z) = \frac{2}{z} - \frac{4}{3} \left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{3} \left(1 + \frac{z}{3}\right)^{-1}$$

$$\therefore f(z) = \frac{2}{z} - \frac{4}{3} \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n + \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$

- ii. $|z| > 3$

Rearrangement of function $f(z)$,

$$f(z) = \frac{2}{z} + \frac{4}{z \left(1 - \frac{3}{z}\right)} + \frac{1}{z \left(1 + \frac{3}{z}\right)} = \frac{2}{z} + \frac{4}{z} \left(1 - \frac{3}{z}\right)^{-1} + \frac{1}{z} \left(1 + \frac{3}{z}\right)^{-1}$$

$$\therefore f(z) = \frac{2}{z} + \frac{4}{z} \sum_{n=0}^{\infty} \left(\frac{3}{z}\right)^n + \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{z}\right)^n$$

Que 5.24. Find the Taylor's and Laurent's series which represent

the function $\frac{z^2 - 1}{(z+2)(z+3)}$ when

- i. $|z| < 2$ ii. $2 < |z| < 3$
iii. $|z| > 3$,

AKTU 2021-22, Marks 10

Answer

Let

$$f(z) = \frac{z^2 - 1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

- i. $|z| < 2$

Rearrangement of function $f(z)$,

$$f(z) = 1 + \frac{3}{2} \left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3} \left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n - \frac{8}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$

It is a Taylor's series within a circle $|z| = 2$.

- ii. $2 < |z| < 3$

$$f(z) = 1 + \frac{3}{z} \left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{3} \left(1 + \frac{z}{3}\right)^{-1}$$

Arranging suitably to make the binomial expansion valid for $2 < |z| < 3$

$$= 1 + \frac{3}{z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$

It is a Laurent's series within the annulus $2 < |z| < 3$.

- iii. $|z| > 3$

$$f(z) = 1 + \frac{3}{z} \left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{z} \left(1 + \frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{z}\right)^n$$

It is a Laurent's series within the annulus $3 < |z| < R$, where R is large.

Que 5.25. Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in Laurent series valid for

- i. $|z-1| > 1$ and ii. $0 < |z-2| < 1$.

Answer

$$f(z) = \frac{z}{(z-1)(2-z)}$$

Using partial fraction,

$$\frac{z}{(z-1)(2-z)} = \frac{A}{(z-1)} + \frac{B}{(2-z)}$$

After solving this, we get

$$A = 1, B = 2$$

$$f(z) = \frac{z}{(z-1)(2-z)} = \frac{1}{(z-1)} + \frac{2}{2-z}$$

$$f(z) = \frac{1}{z-1} - \frac{2}{z-2}$$

i. $|z-1| > 1$

Rearrangement of function $f(z)$,

$$f(z) = \frac{1}{z-1} - \frac{2}{(z-1)-1} = \frac{1}{z-1} - \frac{2}{z-1} \left[1 - \frac{1}{z-1} \right]^{-1}$$

$$= \frac{1}{z-1} - \frac{2}{z-1} \left[1 + \frac{1}{z-1} + \frac{1}{(z-1)^2} + \dots \right]$$

$$f(z) = \frac{1}{z-1} - 2 \sum_{n=0}^{\infty} \frac{1}{(z-1)^{n+1}}$$

ii. $0 < |z-2| < 1$

$$f(z) = \frac{1}{(z-2)+1} - \frac{2}{z-2} = [1 + (z-2)]^{-1} - \frac{2}{z-2}$$

$$f(z) = [1 - (z-2) + (z-2)^2 - (z-2)^3 + \dots] - \frac{2}{z-2}$$

$$f(z) = -\left(\frac{2}{z-2}\right) + \sum_{n=0}^{\infty} (-1)^n (z-2)^n$$

Que 5.26. Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in regions

i. $1 < |z| < 2$

ii. $2 < |z|$

Answer

$$\text{Here, } f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

i. $1 < |z| < 2$,

$$\therefore f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n - \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

This is a series in positive and negative power of z , so it is an expansion of $f(z)$ in Laurent's series within the annulus $1 < |z| < 2$.

ii. $|z| > 2$,

$$\therefore f(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n - \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

This is Laurent's series within the annulus $2 < |z| < R$, where R is large.

Que 5.27. Find the Laurent series expansion of

$$f(z) = \frac{7z-2}{z(z+1)(z+2)} \text{ in the region } 1 < |z+1| < 2.$$

AKTU 2016-17 (III), Marks 05

Answer

Same as Q. 5.25, Page 5-23F, Unit-5.

$$\left(\text{Answer : } f(z) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(z+1)^{n+1}} + \frac{9}{z+1} - 8 \sum_{n=0}^{\infty} \frac{(-1)^n}{(z+1)^{n+1}} \right)$$

PART-4

Singularities, and its Classification, Zeros of Analytic Functions.

CONCEPT OUTLINE

Singularity : A singularity of a function $f(z)$ is a point at which the function ceases to be analytic.

Types of Singularities :

i. **Isolated Singularity :** If $z = a$ is a singularity of $f(z)$ such that $f(z)$ is analytic at each point in its neighbourhood (i.e., there exists a circle with centre a which has no other singularity), then $z = a$ is called an isolated singularity.

In such a case, $f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots (1)$$

For example, $f(z) = \cot(\pi/z)$ is not analytic where as $\tan(\pi/z) = 0$ i.e., at the points $\pi/z = 4\pi$ or $z = 1/n$ ($n = 1, 2, 3, \dots$).

Thus $z = 1, 1/2, 1/3, \dots$ are all isolated singularities as there is no other singularity in their neighbourhood.

But when n is large, $z = 0$ is such a singularity that there are infinite number of other singularities in its neighbourhood.

Thus $z = 0$ is the non-isolated singularity of $f(z)$.

ii. **Removable singularity**: If all the negative powers of $(z-a)$ in

eq. (1) are zero, then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$. Here the singularity

can be removed by defining $f(z)$ at $z = a$ in such a way that it becomes analytic at $z = a$. Such a singularity is called removable singularity.

Thus if $\lim_{z \rightarrow a}$ exists finitely, then $z = a$ is a removable singularity.

iii. **Poles**: If all the negative powers of $(z-a)$ in eq. (1) after the n^{th} are missing, then the singularity at $z = a$ is called a pole of order n . A pole of first order is called a simple pole.

iv. **Essential Singularity**: If the number of negative powers of $(z-a)$ in eq. (1) is infinite, then $z = a$ is called an essential singularity. In this case, $\lim_{z \rightarrow a} f(z)$ does not exist.

Zeros of an Analytic Function: A zero of an analytic function $f(z)$ is that value for z for which $f(z) = 0$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.28. Find the nature and location of singularities of the following functions:

i. $\frac{z - \sin z}{z^3}$

ii. $(z+1) \sin \frac{1}{z-2}$

iii. $\frac{1}{\cos z - \sin z}$

Answer

i. Here, $z = 0$ is a singularity.

Also, $\frac{z - \sin z}{z^3} = \frac{1}{z^3} \left\{ z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \right) \right\} = \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \dots$

Since there are no negative powers of z in the expansion, $z = 0$ is a removable singularity.

ii. $(z+1) \sin \frac{1}{z-2} = (t+2+1) \sin \frac{1}{t}$ Where, $t = z-2$

$$\begin{aligned} &= (t+3) \left\{ \frac{1}{t} - \frac{1}{3!t^3} + \frac{1}{5!t^5} - \dots \right\} \\ &= \left(1 - \frac{1}{3!t^2} + \frac{1}{5!t^4} - \dots \right) + \left(3 - \frac{1}{2t^3} + \frac{3}{5!t^5} - \dots \right) \\ &= 1 + \frac{3}{t} - \frac{1}{6t^2} - \frac{1}{2t^3} + \frac{1}{120t^4} - \dots \\ &= 1 + \frac{3}{z-2} - \frac{1}{6(z-2)^2} - \frac{1}{2(z-2)^3} + \dots \end{aligned}$$

Since there are infinite number of terms in the negative powers of $(z-2)$, $z = 2$ is an essential singularity.

iii. Poles of $f(z) = \frac{1}{\cos z - \sin z}$ are given by equating the denominator to zero, i.e., $\cos z - \sin z = 0$ or $\tan z = 1$ or $z = \pi/4$. Clearly $z = \pi/4$ is a simple pole of $f(z)$.

PART-5

Residues, Cauchy's Residue Theorem and its Application.

CONCEPT OUTLINE

Residues: The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity is called the residue of $f(z)$ at that point. Thus in the Laurent's series expansion of $f(z)$ around $z = a$ i.e., $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots$, the residue of $f(z)$ at $z = a$ is a_{-1} .

$$\text{Res } f(a) = \frac{1}{2\pi i} \oint_C f(z) dz$$

$$\text{i.e., } \oint_C f(z) dz = 2\pi i \text{ Res } f(a)$$

Cauchy's Residue Theorem or Theorem of Residues :

If a function $f(z)$ is analytic, except at a finite number of poles within a closed contour C and continuous on the boundary C , then

$$\oint_C f(z) dz = 2\pi i \sum R$$

$$= 2\pi i \left\{ \begin{array}{l} \text{Sum of residues of } f(z) \text{ at its} \\ \text{poles within } C \end{array} \right\}$$

Evaluation of Real Integral of Type

$$\int_0^{2\pi} f(\cos \theta, \sin \theta) d\theta, \quad \int_{-1}^1 f(x) dx :$$

Integrals of the type $\int_0^{2\pi} f(\cos \theta, \sin \theta) d\theta$, where $f(\cos \theta, \sin \theta)$ is a rational function of $\cos \theta$ and $\sin \theta$.

$$\int_0^{2\pi} f(\cos \theta, \sin \theta) d\theta = \oint_C f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \frac{dz}{iz}$$

where C is a unit circle of $|z| = 1$.

Questions-Answers**Long Answer Type and Medium Answer Type Questions**

Que 5.29. Find the residues of $f(z) = \frac{z-3}{z^2+2z+5}$ at its poles. Hence

or otherwise evaluate $\oint_C \frac{z-3}{z^2+2z+5}$, where C is the circle $|z+1-i| = 2$.

AKTU 2012-13 (IV), Marks 05

Answer

The poles of $f(z) = \frac{z-3}{z^2+2z+5}$ are given by

$$z^2+2z+5=0 \Rightarrow z = -1 \pm 2i$$

Only the pole $z = -1 + 2i$ lies inside the circle $|z+1-i| = 2$

Residue of $f(z)$ at $z = -1 + 2i$ is

$$= \lim_{z \rightarrow -1+2i} (z+1-2i) f(z)$$

$$= \lim_{z \rightarrow \alpha} \frac{(z-\alpha)(z-3)}{z^2+2z+5}, \text{ where } \alpha = -1+2i \left(\text{Form } \frac{0}{0} \right)$$

$$= \lim_{z \rightarrow \alpha} \frac{(z-\alpha)+(z-3)}{2z+2} \quad (\text{By L' Hospital's Rule})$$

$$= \frac{\alpha-3}{2\alpha+2} = \frac{-1+2i-3}{-2+4i+2} = \frac{i-2}{2i}$$

$\therefore$ By Cauchy's residue theorem,

$$\oint_C \frac{z-3}{z^2+2z+5} dz = 2\pi i \left(\frac{i-2}{2i} \right) = \pi(i-2)$$

Que 5.30. Determine the poles and residues at each pole for

$f(z) = \frac{z-1}{(z+1)^2(z-2)}$ and hence evaluate $\oint_C f(z) dz$ where C is the circle

$$|z-i| = 2.$$

AKTU 2013-14 (IV), Marks 05

Answer

Poles of $f(z)$ are given by

$$(z+1)^2(z-2) = 0, z = -1 \text{ (Order 2), } 2 \text{ (Simple pole)}$$

Residue of $f(z)$ at $z = -1$ is,

$$R_1 = \frac{1}{(2-1)!} \left[\frac{d}{dz} \left\{ (z+1)^2 \frac{z-1}{(z+1)^2(z-2)} \right\} \right]_{z=-1}$$

$$= \left[\frac{d}{dz} \left(\frac{z-1}{z-2} \right) \right]_{z=-1} = \left[\frac{-1}{(z-2)^2} \right]_{z=-1} = \frac{-1}{9}$$

Residue of $f(z)$ at $z = 2$ is,

$$R_2 = \lim_{z \rightarrow 2} (z-2) \frac{z-1}{(z+1)^2(z-2)} = \lim_{z \rightarrow 2} \frac{z-1}{(z+1)^2} = \frac{1}{9}$$

The given curve $C = |z-i| = 2$ is a circle whose centre is at $z = i$ [i.e., at (0, 1)] and radius is 2. Clearly, only the pole $z = -1$ lies inside the curve C .

Hence, by Cauchy's residue theorem

$$\oint_C f(z) dz = 2\pi i (R_1) = 2\pi i \left(\frac{-1}{9} \right) = -\frac{2\pi i}{9}$$

Que 5.31. Evaluate $\int_C \frac{e^z}{\cos \pi z} dz$, where C is the unit circle $|z| = 1$.

Answer

$$\text{Here, } f(z) = \frac{e^z}{\cos \pi z} = \frac{e^z}{1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \dots}$$

It has simple poles at $z = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \dots$, of which only $z = \pm \frac{1}{2}$ lie inside the circle $|z| = 1$.

Residue of $f(z)$ at $z = \frac{1}{2}$ is

$$\begin{aligned} \lim_{z \rightarrow \frac{1}{2}} \left(z - \frac{1}{2}\right) f(z) &= \lim_{z \rightarrow \frac{1}{2}} \frac{\left(z - \frac{1}{2}\right) e^z}{\cos \pi z} \quad \left[\text{Form } \frac{0}{0}\right] \\ &= \lim_{z \rightarrow \frac{1}{2}} \frac{\left(z - \frac{1}{2}\right) e^z}{-\pi \sin \pi z} \quad [\text{By L' Hospital's Rule}] \\ &= \frac{e^{1/2}}{-\pi} \end{aligned}$$

Similarly, residue of $f(z)$ at $z = -\frac{1}{2}$ is $\frac{e^{-1/2}}{\pi}$

∴ By residue theorem,

$$\begin{aligned} \oint_C \frac{e^z}{\cos \pi z} dz &= 2\pi i \quad (\text{Sum of residues}) \\ &= 2\pi i \left(-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi} \right) = -4i \left(\frac{e^{1/2} - e^{-1/2}}{2} \right) = -4i \sinh \frac{1}{2} \end{aligned}$$

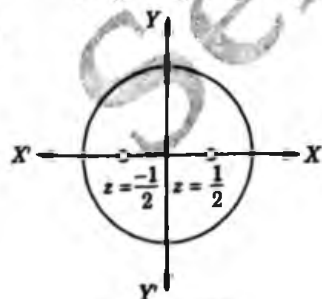


Fig. 5.31.1.

Que 5.32. Using calculus of residue, evaluate the following integral

$$\int_0^{2\pi} \frac{dx}{(a^2 + x^2)^2}$$

AKTU 2015-16 (IV), Marks 10

Answer

Let,

$$I = \int_0^{2\pi} \frac{dx}{(x^2 + a^2)^2}$$

$$f(x) = \frac{1}{(x^2 + a^2)^2}$$

Poles, $(x^2 + a^2)^2 = 0$

$$\Rightarrow x^2 + a^2 = 0$$

$$\therefore x = \pm ai$$

Only one pole but repeated in nature.

Residue at $x = ai$

$$\begin{aligned} &= \frac{1}{(2-1)!} \left[\frac{d}{dx} \left\{ (x-ai)^2 \times \frac{1}{(x-ai)^2(x+ai)^2} \right\} \right]_{x=ai} \\ &= \frac{1}{(2-1)!} \left[\frac{d}{dx} \left(\frac{1}{(x+ai)^2} \right) \right]_{x=ai} = \left[\frac{-2}{(x+ai)^3} \right]_{x=ai} \\ &= \frac{-1}{-4a^3 i} = \frac{1}{4a^3 i} \end{aligned}$$

Using Cauchy's Residue theorem,

$$\int f(z) dz = 2\pi i \quad (\text{Sum of residue})$$

$$\int_0^{2\pi} \frac{1}{(x^2 + a^2)^2} dx = 2\pi i \times \frac{1}{4a^3 i} = \frac{\pi}{2a^3}$$

Using property of integration,

$$\int_0^{2\pi} \frac{1}{(x^2 + a^2)^2} dx = \frac{\pi}{4a^3}$$

Que 5.33. Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5 + 4 \cos \theta} d\theta$.

AKTU 2013-14 (IV), Marks 05

Answer

$$\text{Let } I = \int_0^{2\pi} \frac{\cos 3\theta}{5 + 4 \cos \theta} d\theta = \text{Real part of } \int_0^{2\pi} \frac{e^{i3\theta}}{5 + 2(e^{\theta} + e^{-\theta})} d\theta$$

$$= \text{Real part of } \oint_C \left| \frac{z^3}{5 + 2\left(z + \frac{1}{z}\right)} \right| \frac{dz}{iz} \quad \left(\text{Writing } e^{\theta} = z, d\theta = \frac{dz}{iz} \right)$$

$$= \text{Real part of } \frac{1}{i} \int_C \frac{z^3}{(2z+1)(z+2)} dz$$

Singularities of $f(z)$ are given by, $(2z+1)(z+2) = 0$

$$z = -\frac{1}{2}, -2$$

Only, $z = -\frac{1}{2}$ lies within the unit circle $|z| = 1$.

$$\begin{aligned} \therefore \text{Residue of } f(z) \left(\text{at } z = -\frac{1}{2} \right) &= \lim_{z \rightarrow -\frac{1}{2}} \left(z + \frac{1}{2} \right) \times \frac{z^3}{i(2z+1)(z+2)} \\ &= \lim_{z \rightarrow -\frac{1}{2}} \frac{z^3}{2i(z+2)} = \frac{1}{2i} \left(\frac{-1}{8} \right) \times \left(\frac{2}{3} \right) = \frac{-1}{24i} \end{aligned}$$

Hence by Cauchy's Residue theorem

$$I = \oint_C f(z) dz = 2\pi i \left(\frac{-1}{24i} \right) = -\frac{\pi}{12}$$

Que 5.34. Evaluate by contour integration: $\int_0^{2\pi} e^{-\cos \theta} \cos(n\theta + \sin \theta) d\theta$; $n \in \mathbb{I}$.

AKTU 2021-22, Marks 10

Answer

Consider the integral

$$\begin{aligned} I &= \int_0^{2\pi} e^{-\cos \theta} [\cos(n\theta + \sin \theta) - i \sin(n\theta + \sin \theta)] d\theta \\ &= \int_0^{2\pi} e^{-\cos \theta} e^{-i(\sin \theta + n\theta)} d\theta \\ &= \int_0^{2\pi} e^{-(\cos \theta + i \sin \theta)} e^{-in\theta} d\theta \\ \int_0^{2\pi} e^{-e^{i\theta}} e^{-in\theta} d\theta &= \int_C \left(e^{-z} \frac{1}{z^n} \right) \frac{dz}{iz} \end{aligned}$$

Writing $e^{i\theta} = z$, $d\theta = \frac{dz}{iz}$ where, C denotes the unit circle $|z| = 1$.

$$\frac{1}{i} \int_C \frac{e^{-z}}{z^{n+1}} dz = \int_C f(z) dz \quad \text{where } f(z) = \frac{e^{-z}}{i z^{n+1}}$$

$$= 2\pi i \sum R^* \text{ By Cauchy's residue theorem.}$$

Obviously the only pole of $f(z)$ within the contour

C is $z = 0$, of order $n+1$.

At $z = 0$, the residue

$$= \frac{1}{n!} \left\{ \frac{d^n}{dz^n} \left(\frac{e^{-z}}{i} \right) \right\}_{z=0} = \frac{(-1)^n}{i(n)!} = \sum R^*.$$

$$\therefore I = 2\pi i \times \frac{(-1)^n}{i(n)!} = \frac{2\pi}{n!} (-1)^n$$

$$\text{i.e., } \int_0^{2\pi} e^{-\cos \theta} [\cos(n\theta + \sin \theta) - i \sin(n\theta + \sin \theta)] d\theta$$

$$= \frac{2\pi}{n!} (-1)^n$$

Equating real parts, we have

$$\int_0^{2\pi} e^{-\cos \theta} \cos(n\theta + \sin \theta) d\theta = \frac{2\pi}{n!} (-1)^n$$

Que 5.35. Evaluate: $\int_0^{2\pi} \frac{d\theta}{a + b \sin \theta}$ if $a > |b|$

AKTU 2016-17 (IV), Marks 05

Answer

Consider the integration round a unit circle $|z| = 1$

So that $z = e^{i\theta} \therefore d\theta = \frac{dz}{iz}$

$$\text{Also, } \sin \theta = \frac{1}{2i} (e^{\theta} - e^{-\theta}) = \frac{1}{2i} \left(z - \frac{1}{z} \right)$$

Then the given integral reduces to

$$\begin{aligned} I &= \oint_C \frac{1}{a + b \left(\frac{z - \frac{1}{z}}{2i} \right)} \left(\frac{dz}{iz} \right) = \oint_C \frac{2iz}{bz^2 + 2iaz - b} \left(\frac{dz}{iz} \right) \\ &= \frac{2}{b} \oint_C \frac{dz}{z^2 + \frac{2ia}{b}z - 1} \end{aligned}$$

Poles are given by, $z^2 + \frac{2ia}{b}z - 1 = 0$

$$\begin{aligned} z &= \frac{-\frac{2ia}{b} \pm \sqrt{\frac{4a^2}{b^2} + 4}}{2} = \frac{-ia}{b} \pm \frac{\sqrt{b^2 - a^2}}{b} \\ &= \frac{-ia}{b} \pm \frac{i\sqrt{a^2 - b^2}}{b} = \alpha, \beta \text{ (simple poles)} \end{aligned}$$

where, $\alpha = \frac{-ia}{b} + \frac{i\sqrt{a^2 - b^2}}{b}$ and $\beta = \frac{-ia}{b} - \frac{i\sqrt{a^2 - b^2}}{b}$

Clearly, $|\beta| > 1$

But $\alpha\beta = -1$

$|\alpha\beta| = 1$

$|\alpha||\beta| = 1$

$|\alpha| < 1$

Hence $z = \alpha$ is the only pole which lies inside circle $C = |z| = 1$.

Residue of $f(z)$ at $(z = \alpha)$ is

$$R = \lim_{z \rightarrow \alpha} (z - \alpha) \times \frac{2}{b(z - \alpha)(z - \beta)} = \frac{2}{b(\alpha - \beta)}$$

$$= \frac{2}{b \left(\frac{2i\sqrt{a^2 - b^2}}{b} \right)} = \frac{1}{i\sqrt{a^2 - b^2}}$$

$\therefore$ By Cauchy's Residue theorem,

$$I = 2\pi i(R) = 2\pi i \left(\frac{1}{i\sqrt{a^2 - b^2}} \right) = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a + b \sin \theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Que 5.36. Apply residue theorem to evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)}$.

AKTU 2018-19, Marks 10

Answer

We consider $\int_C \frac{z^2 dz}{(z^2 + 1)(z^2 + 4)} = \int_C f(z) dz$

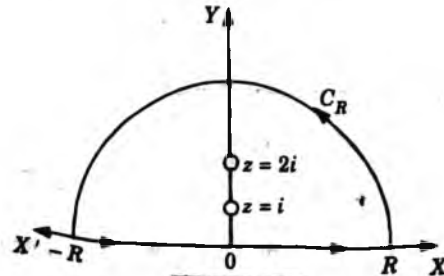


Fig. 5.36.1.

Where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to $+R$.

The integral has simple poles at

$$z = \pm i, z = \pm 2i$$

of which $z = i, 2i$ only lie inside C .

$$\text{The residue (at } z = i) = \lim_{z \rightarrow i} (z - i) \frac{z^2}{(z + i)(z - i)(z^2 + 4)}$$

$$= \lim_{z \rightarrow i} \frac{z^2}{(z + i)(z^2 + 4)} = \frac{-1}{2i(-1 + 4)} = \frac{-1}{6i}$$

$$\text{The residue (at } z = 2i) = \lim_{z \rightarrow 2i} (z - 2i) \frac{z^2}{(z^2 + 1)(z + 2i)(z - 2i)}$$

$$= \lim_{z \rightarrow 2i} \frac{z^2}{(z^2 + 1)(z + 2i)} = \frac{(2i)^2}{(-4 + 1)(2i + 2i)} = \frac{1}{3}$$

By theorem of residue,

$$\int_C f(z) dz = 2\pi i [\text{Res } f(i) + \text{Res } f(2i)] = 2\pi i \left(-\frac{1}{6i} + \frac{1}{3i} \right) = \frac{\pi}{3}$$

$$\text{i.e., } \int_{-R}^R f(x) dx + \int_{C_R} f(z) dz = \frac{\pi}{3} \quad (5.36.1)$$

Hence by making $R \rightarrow \infty$, eq. (5.36.1) becomes

$$\int_{-\infty}^{\infty} f(x) dx + \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = \frac{\pi}{3}$$

Now $R \rightarrow \infty$, $\int_{C_R} f(z) dz$ vanishes,

For any point on C_R as $|z| \rightarrow \infty$, $f(z) = 0$

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0, \int_{-\infty}^{\infty} f(x) dx = \frac{\pi}{3}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)} = \frac{\pi}{3}$$





Ordinary Differential Equation of Higher Order

(2 Marks Questions)

- 1.1. Find the order and degree of the following differential equation

$$\frac{d^2 y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx}\right)^3} = 0$$

Also explain your answer.

AKTU 2015-16, Marks 02

$$\frac{d^2 y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx}\right)^3} = 0$$

On rearranging,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^3} = -\frac{d^2 y}{dx^2}$$

On squaring both side,

$$1 + \left(\frac{dy}{dx}\right)^3 = \left(\frac{d^2 y}{dx^2}\right)^2$$

Again rearranging the equation, we have

$$\left(\frac{d^2 y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^3 - 1 = 0$$

∴ Order = 2 and degree = 2

- 1.2. Find the roots of the auxiliary equation of the differential equation :

$$\frac{d^2 y}{dt^2} - 6 \frac{dy}{dt} + 9y = 4e^{3t}$$

AKTU 2015-16, Marks 02

Differential equation is,

$$\frac{d^2 y}{dt^2} - 6 \frac{dy}{dt} + 9y = 4e^{3t}$$

$$D^2 y - 6Dy + 9y = 4e^{3t}$$

Auxiliary equation,

$$m^2 - 6m + 9 = 0$$

$$(m - 3)^2 = 0$$

$$m = 3, 3$$

1.3. Solve $\frac{d^2 y}{dt^2} - 3 \frac{dy}{dt} + 4y = 0$.

AKTU 2012-13, Marks 02

Auxiliary equation is,
 $m^2 - 3m + 4 = 0$

$$m = \frac{-(-3) \pm \sqrt{9 - 16}}{2}$$

$$m = \frac{3 \pm i\sqrt{7}}{2}$$

$$m = \frac{3}{2} \pm \frac{\sqrt{7}}{2} i$$

Since, roots of auxiliary equation are complex, then

$$CF = e^{\frac{3}{2}x} \left(C_1 \cos \frac{\sqrt{7}}{2} x + C_2 \sin \frac{\sqrt{7}}{2} x \right)$$

and $PI = 0$

Therefore, complete solution = CF + PI

$$= e^{\frac{3}{2}x} \left(C_1 \cos \frac{\sqrt{7}}{2} x + C_2 \sin \frac{\sqrt{7}}{2} x \right)$$

1.4. Find the PI of $\frac{d^2 y}{dx^2} + 4y = \sin 2x$.

AKTU 2018-19 (II), Marks 02

Ans

$$PI = \frac{1}{D^2 + 4} \sin 2x = \frac{1}{(2)^2 + 4} \sin 2x$$

$$= \frac{1}{2D} \sin 2x$$

$$= \frac{-x}{4} \cos 2x$$

1.5. Find the general solution of $(2D + 1)^2 y = 0$, where $D = \frac{d}{dt}$.

AKTU 2011-12, Marks 02

Ans

$$(2D + 1)^2 y = 0$$

Auxiliary equation is,

$$(2m + 1)^2 = 0$$

$$m = -\frac{1}{2}, -\frac{1}{2}$$

General solution is $y = (C_1 + C_2 t)e^{-t/2}$

1.6. Solve simultaneous equation $\frac{dx}{dt} = 3y$, $\frac{dy}{dt} = 3x$.

AKTU 2018-19 (II), Marks 02

$$\frac{dx}{dt} = 3y \Rightarrow Dx - 3y = 0 \quad \dots(1.6.1)$$

$$\frac{dy}{dt} = 3x \Rightarrow -3x + Dy = 0 \quad \dots(1.6.2)$$

Multiplying eq. (1.6.1) by D and eq. (1.6.2) by 3 and adding the two equations, we have

$$(D^2 - 3^2)x = 0$$

Auxiliary equation,

$$m^2 - 3^2 = 0$$

$$m = \pm 3$$

$$CF = C_1 e^{3t} + C_2 e^{-3t}$$

$$PI = 0$$

$$x = C_1 e^{3t} + C_2 e^{-3t}$$

From eq. (1.6.1)

$$y = \frac{1}{3} \frac{dx}{dt} = \frac{1}{3} \frac{d}{dt} (C_1 e^{3t} + C_2 e^{-3t})$$

$$= \frac{1}{3} (3C_1 e^{3t} - 3C_2 e^{-3t})$$

$$= C_1 e^{3t} - C_2 e^{-3t}$$

1.7. Solve: $(2D - 1)^2 y = 0$.

AKTU (2D - 1)^2 y = 0

Auxiliary equation is,

$$(2m - 1)^2 = 0$$

$$m = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$$

$$y = (C_1 + xC_2 + x^2 C_3) e^{x/2}$$

AKTU 2014-15, Marks 02

1.8. Find the particular integral of $\frac{d^2 y}{dx^2} = x^2 + 2x - 1$.

AKTU 2012-13, Marks 02

$$PI = \frac{1}{D^2} (x^2 + 2x - 1)$$

$$= \frac{1}{1 + D^2 - 1} (x^2 + 2x - 1)$$

$$= \frac{1}{[1 + (D^2 - 1)]} (x^2 + 2x - 1)$$

$$\begin{aligned} &= [1 + (D^2 - 1)]^{-1} (x^2 + 2x - 1) \\ &= [1 - (D^2 - 1)] (x^2 + 2x - 1) \\ &= (2 - D^2) (x^2 + 2x - 1) \\ &= 2x^2 + 4x - 2 - D^2 (x^2 + 2x - 1) \\ &= 2x^2 + 4x - 2 - 2 \\ &= 2x^2 + 4x - 4 \\ &= 2(x^2 + 2x - 2) \end{aligned}$$

1.9. For the differential equation $x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x$, find its complementary function.

AKTU Putting $x = e^t$
i.e., $t = \ln x$, the given equation becomes

$$[D(D-1) + 4D + 2]y = e^t$$

$$\text{i.e., } (D^2 + 3D + 2)y = e^t$$

Auxiliary equation is,

$$m^2 + 3m + 2 = 0$$

$$m = -1, -2$$

$$CF = C_1 e^{-t} + C_2 e^{-2t} = C_1 x^{-1} + C_2 x^{-2} = \frac{C_1}{x} + \frac{C_2}{x^2}$$

1.10. Solve the differential equation $\frac{d^2 y}{dx^2} = -12x^2 + 24x - 20$ with the condition $x = 0, y = 5$ and $x = 2, y = 21$ and hence find the value of y at $x = 1$.

AKTU 2016-17, Marks 02

AKTU Given: $\frac{d^2 y}{dx^2} = -12x^2 + 24x - 20$

On integrating the above equation, we get

$$\frac{dy}{dx} = -4x^3 + 12x^2 - 20x + C_1 \quad \dots(1.10.1)$$

Again integrating eq. (1.10.1), we have

$$y = -x^4 + 4x^3 - 10x^2 + C_1 x + C_2 \quad \dots(1.10.2)$$

At $x = 0, y = 5$

$\therefore$ From eq. (1.10.2),

$$C_2 = 5$$

At $x = 2, y = 21$

$\therefore$ From eq. (1.10.2),

$$21 = -16 + 32 - 40 + 2C_1 + 5$$

$$21 = -19 + 2C_1$$

$$2C_1 = 40$$

$$C_1 = 20$$

Putting the value of C_1 and C_2 in eq. (1.10.2), we get

$$y = -x^4 + 4x^3 - 10x^2 + 20x + 5$$

At $x = 1$; $y = -1 + 4 - 10 + 20 + 5$

$$y = 18$$

- 1.11. Find the differential equation which represents the family of straight lines passing through the origins.

AKTU 2021-22 (II), Marks 02

Ans. General equation for straight line is,

$$(y - y_1) = m(x - x_1)$$

For a line passing through origin $x_1 = 0$, and $y_1 = 0$

So, line passing through origin has equation

$$y = mx$$

...(1.11.1)

Differentiate eq. (1.11.1) with respect to x

$$\frac{dy}{dx} = \frac{y}{x} \text{ is required equation}$$

- 1.12. State the criterion for linearly independent solutions of the homogeneous linear n^{th} order differential equation.

AKTU 2021-22 (II), Marks 02

Ans. Let $y_1, y_2, \dots, y_n$ be n solution of the homogenous linear n^{th} order differential equation on an interval I . Then the set of solution is linearly independent on I if and only if $W(y_1, y_2, \dots, y_n) \neq 0$ for every x in the interval.

- 1.13. For a differential equation $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + y = 0$, find the value of a for which the differential equation characteristic equation has equal number of roots.

AKTU 2016-17, Marks 02

Ans. The characteristic equation of the given differential equation is

$$m^2 + 2am + 1 = 0$$

For equal roots,

$$(2a)^2 - 4 \times 1 \times 1 = 0$$

$$4a^2 - 4 = 0$$

$$4a^2 = 4$$

$$a^2 = 1$$

$$a = \pm 1$$

- 1.14. Solve: $(D + 1)^3 y = 2e^{-x}$.

AKTU 2017-18, Marks 02

Ans. Auxiliary equation is

$$(m + 1)^3 = 0$$

$$m = -1, -1, -1$$

$$CF = (C_1 + C_2x + C_3x^2)e^{-x}$$

$$PI = \frac{1}{(D + 1)^3} 2e^{-x}$$

$$= x \frac{1}{3(D + 1)^2} 2e^{-x}$$

$$= x^2 \frac{1}{3 \times 2(D + 1)} 2e^{-x}$$

$$= x^3 \frac{1}{3 \times 2 \times 1} 2e^{-x} = \frac{x^3}{3} e^{-x}$$

$$y = CF + PI$$

$$= (C_1 + C_2x + C_3x^2)e^{-x} + \frac{x^3}{3} e^{-x}$$

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Laplace Transform (2 Marks Questions)

2.1. Define Laplace transform

Ans. Let $f(t)$ be function defined for all positive values of $t(t \geq 0)$, then

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

provided the integral exists, is called the Laplace transform of $f(t)$. It is denoted as

$$L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

2.2. Find the Laplace Transform of e^{at} .

$$\begin{aligned} \text{Ans. } L(e^{at}) &= \int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{-s+at} dt \\ &= \int_0^{\infty} e^{(-s+a)t} dt = \int_0^{\infty} e^{-(s-a)t} dt \\ &= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} = -\frac{1}{s-a} \left[\frac{1}{e^{(s-a)t}} \right]_0^{\infty} \\ &= \frac{-1}{(s-a)} (0-1) = \frac{1}{s-a} \end{aligned}$$

2.3. Determine the Laplace transform of $\cos at$.

$$\begin{aligned} \text{Ans. } L(\cos at) &= L\left(\frac{e^{iat} + e^{-iat}}{2}\right) \left[\because \cos at = \frac{e^{iat} + e^{-iat}}{2} \right] \\ &= \frac{1}{2} [L(e^{iat} + e^{-iat})] = \frac{1}{2} [L(e^{iat}) + L(e^{-iat})] \\ &= \frac{1}{2} \left[\frac{1}{s-ia} + \frac{1}{s+ia} \right] = \frac{1}{2} \left[\frac{s+ia + s-ia}{s^2 + a^2} \right] \\ &= \frac{s}{s^2 + a^2} \end{aligned}$$

2.4. Find the Laplace transform of $t^{-1/2}$.

Ans. We know that,

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$\text{Put } n = -\frac{1}{2}, L(t^{-1/2}) = \frac{\frac{-1/2+1}{2}!}{s^{-1/2+1}} = \frac{\frac{1}{2}!}{\sqrt{s}} = \frac{\sqrt{\pi}}{\sqrt{s}} \quad \left(\text{where, } \frac{1}{2}! = \sqrt{\pi} \right)$$

2.5. Find the Laplace transform of $\cos^2 t$.

Ans. We know that,

$$\cos 2t = 2 \cos^2 t - 1$$

$$\cos^2 t = \frac{1}{2} [\cos 2t + 1]$$

$$\begin{aligned} L(\cos^2 t) &= L\left[\frac{1}{2} (\cos 2t + 1)\right] = \frac{1}{2} [L(\cos 2t) + L(1)] \\ &= \frac{1}{2} \left[\frac{s}{s^2 + (2)^2} + \frac{1}{s} \right] = \frac{1}{2} \left[\frac{s}{s^2 + 4} + \frac{1}{s} \right] \end{aligned}$$

2.6. Write the change of scale property of Laplace transform.

Ans. If $L\{f(t)\} = F(s)$

$$\text{Then } L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

2.7. Find the Laplace transform of $e^{-3t} (2 \cos 5t - 3 \sin 5t)$

$$\begin{aligned} \text{Ans. } L(e^{-3t} (2 \cos 5t - 3 \sin 5t)) &= 2L(e^{-3t} \cos 5t) - 3L(e^{-3t} \sin 5t) \\ &= 2 \frac{s+3}{(s+3)^2 + 5^2} - 3 \frac{5}{(s+3)^2 + 5^2} = \frac{2s-9}{s^2 + 6s + 34} \end{aligned}$$

2.8. Find $L\left\{\frac{\sin at}{t}\right\}$, given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left\{\frac{1}{s}\right\}$.

Ans. By the change of scale property,

$$\begin{aligned} L\left\{\frac{\sin at}{t}\right\} &= \frac{1}{a} \tan^{-1}\left\{\frac{1}{s/a}\right\} \\ &= \frac{1}{a} \tan^{-1}\left(\frac{a}{s}\right) \text{ i.e., } L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left(\frac{a}{s}\right) \end{aligned}$$

2.9. Find the Laplace transform of $t \cos at$.

Ans. Since $L(\cos at) = s/(s^2 + a^2)$

$$\therefore L(t \cos at) = -\frac{d}{ds} \left(\frac{s}{s^2 + a^2} \right) = -\frac{s^2 + a^2 - s2s}{(s^2 + a^2)^2}$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

2.10. Find the Laplace transform of $f(t) = \int_0^t \frac{\sin at}{t} dt$.

Ans

$$L(\sin at) = \frac{a}{s^2 + a^2}$$

$$\begin{aligned} L\left(\frac{\sin at}{t}\right) &= \int_s^\infty \frac{a}{s^2 + a^2} ds = \left[\tan^{-1} \frac{s}{a} \right]_s^\infty \\ &= \frac{\pi}{2} - \tan^{-1} \left(\frac{s}{a} \right) = \cot^{-1} \left(\frac{s}{a} \right) \end{aligned}$$

$$\text{Hence, } L\left[\int_0^t \frac{\sin at}{t} dt\right] = \frac{1}{s} \cos^{-1} \left(\frac{s}{a} \right)$$

2.11. Define unit step function.

Ans Unit step function: The unit step functions $u(t-a)$ is defined as follows:

$$u(t-a) = \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t \geq a \end{cases} \quad (\text{where, } a \geq 0)$$



Fig. 2.11.1.

2.12. What is the Laplace transform of periodic function?

Ans Let $f(t)$ be a periodic function with period T , then

$$L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$

2.13. Determine the inverse Laplace transform of $\frac{1}{s-2}$.

Ans

$$L^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$$

2.14. Show that $\frac{1}{s^{1/2}} = L\left[\frac{1}{\sqrt{\pi t}}\right]$.

Ans We have to show that $\frac{1}{s^{1/2}} = L\left[\frac{1}{\sqrt{\pi t}}\right]$.

$$\text{Now, } L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!} = \frac{t^{n-1}}{\Gamma(n)}$$

$$\text{So, } L^{-1}\left(\frac{1}{s^{1/2}}\right) = \frac{t^{\frac{1}{2}-1}}{\Gamma(\frac{1}{2})} = \frac{t^{-1/2}}{\Gamma(\frac{1}{2})} = \frac{t^{-1/2}}{\sqrt{\pi}}$$

$$\Rightarrow L^{-1}\left(\frac{1}{s^{1/2}}\right) = \frac{1}{\sqrt{\pi t}} \Rightarrow \frac{1}{s^{1/2}} = L\left[\frac{1}{\sqrt{\pi t}}\right]$$



Sequence and Series (2 Marks Questions)

3.1. Examine the sequence $a_n = 2^n$ for convergence.

Ans $\lim_{n \rightarrow \infty} (2^n) = \infty$. Hence the sequence a_n is divergent.

3.2. Test the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$

AKTU 2021-22 (II), Marks 02

Ans Here,

$$u_n = \frac{1}{n} \sin \frac{1}{n} = \frac{1}{n} \left[\frac{1}{n} - \frac{1}{3!n^3} + \frac{1}{5!n^5} \dots \right]$$

$$= \frac{1}{n^2} \left[1 - \frac{1}{3!n^2} + \frac{1}{5!n^4} \dots \right]$$

Taking $v_n = \frac{1}{n^2}$ we have

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \left[1 - \frac{1}{3!n^2} + \frac{1}{5!n^4} \dots \right] = 1 \neq 0$$

Since $\sum v_n$ is convergent,
 $\sum u_n$ is also convergent.

3.3. Write down the statement of Raabe's test.

Ans Raabe's Test : In the positive term series $\sum u_n$, if

$$\lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = k,$$

then the series converges for $k > 1$ and diverges for $k < 1$, but the test fails for $k = 1$.

3.4. Find the constant term when $f(x) = 1 + |x|$ is expanded in fourier series in the interval $(-3, 3)$.

AKTU 2021-22 (II), Marks 02

Ans $f(x) = 1 + |x|$ defined in $(-3, 3)$

$$a_0 = \frac{2}{3} \int_0^3 (1+x) dx = \frac{2}{3} \left[x + \frac{x^2}{2} \right]_0^3$$

$$= \frac{2}{3} \left[3 + \frac{3^2}{2} \right] = \frac{2}{3} \times \frac{15}{2} = 5$$

3.5. Find the Fourier constant a_1 of $f(x) = x^2, -\pi \leq x \leq \pi$.

AKTU 2018-19 (II), Marks 02

Ans We know that,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

$$= \frac{2}{\pi} \left[\left(x^2 \frac{\sin nx}{n} \right)_0^{\pi} - \int_0^{\pi} 2x \frac{\sin nx}{n} dx \right]$$

$$= -\frac{4}{\pi n} \int_0^{\pi} x \sin nx dx$$

$$= -\frac{4}{\pi n} \left[\left(x \left(-\frac{\cos nx}{n} \right) \right)_0^{\pi} - \int_0^{\pi} 1 \left(-\frac{\cos nx}{n} \right) dx \right]$$

$$= \frac{4}{\pi n^2} (\pi \cos n\pi) = \frac{4}{n^2} (-1)^n$$

For $n = 1, a_1 = -4$

3.6. Discuss the convergence of sequence $a_n = \frac{2n}{n^2 + 1}$

AKTU 2018-19 (II), Marks 02

Ans $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{2n}{n^2 + 1} \right) = \lim_{n \rightarrow \infty} \frac{n^2}{n^2} \left(\frac{2/n}{1 + 1/n^2} \right) = 0$

Hence, given sequence is convergent.

3.7. Find the constant term if the function $f(x) = x + x^2$ is expanded in Fourier series defined in $(-1, 1)$.

AKTU 2011-12, Marks 02

Ans $f(x) = x + x^2$

$$a_0 = \frac{1}{1} \int_{-1}^1 (x + x^2) dx = 0 + \left[\frac{x^3}{3} \right]_{-1}^1 = \left(\frac{1+1}{3} \right)$$

$$a_0 = \frac{2}{3}$$

- 3.8. If $f(x) = 1$ is expanded in a Fourier sine series in $(0, \pi)$, then find the value of b_n . **AKTU 2012-13, Marks 02**

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\
 &= \frac{1}{\pi} \int_0^{\pi} 1 \sin nx \, dx = \frac{1}{\pi} \int_0^{\pi} \sin nx \, dx \\
 &= \frac{1}{\pi} \left[\frac{-\cos nx}{n} \right]_0^{\pi} = \frac{1}{n} \left[\frac{-\cos n\pi}{n} + \frac{1}{n} \right] \\
 &= \frac{1}{n\pi} (1 - \cos n\pi)
 \end{aligned}$$

- 3.9. Find the value of the Fourier coefficient a_0 for the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

AKTU 2015-16, Marks 02

Let the Fourier series be,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$a_0 = \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) \, dx + \int_0^{\pi} f(x) \, dx \right]$$

$$a_0 = \frac{1}{\pi} \left[0 + \int_0^{\pi} x \, dx \right]$$

$$a_0 = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$a_0 = \frac{1}{\pi} \left[\frac{\pi^2 - 0}{2} \right] = \frac{\pi}{2}$$

- 3.10. Expand for $f(x) = k$ for $0 < x < 2$ in a half range sine series.

AKTU

$$f(x) = k$$

$$b_n = \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c} \, dx \text{ in half range } (0, c)$$

$$\begin{aligned}
 &= \frac{2}{2} \int_0^2 k \sin \frac{n\pi x}{2} \, dx = k \frac{2}{n\pi} \left(-\cos \frac{n\pi x}{2} \right)_0^2 \\
 &= \frac{2k}{n\pi} [-\cos n\pi + 1]
 \end{aligned}$$

Half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

$$k = \sum_{n=1}^{\infty} \frac{2k}{n\pi} [-\cos n\pi + 1] \sin \frac{n\pi x}{2}$$

- 3.11. Find the Fourier coefficient for the function

$$f(x) = x^2, 0 < x < 2\pi.$$

AKTU 2016-17, Marks 02

The Fourier coefficients for the given function are as follows :

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx = \frac{1}{\pi} \int_0^{2\pi} x^2 \, dx = \frac{1}{\pi} \left[\frac{x^3}{3} \right]_0^{2\pi} = \frac{8\pi^2}{3}$$

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx \, dx \\
 &= \frac{1}{\pi} \left[\left(\frac{x^2 \sin nx}{n} \right)_0^{2\pi} - \int_0^{2\pi} 2x \frac{\sin nx}{n} \, dx \right] \quad (\because \sin n\pi = 0)
 \end{aligned}$$

$$= -\frac{2}{\pi n} \int_0^{2\pi} x \sin nx \, dx$$

$$= -\frac{2}{\pi n} \left[\left\{ x \left(-\frac{\cos nx}{n} \right) \right\}_0^{2\pi} - \int_0^{2\pi} 1 \left(-\frac{\cos nx}{n} \right) \, dx \right]$$

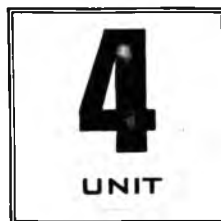
$$= -\frac{2}{\pi n} \left[-\frac{2\pi}{n} - 0 \right] = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\left\{ x^2 \left(-\frac{\cos nx}{n} \right) \right\}_0^{2\pi} - \int_0^{2\pi} 2x \frac{\cos nx}{n} \, dx \right]$$

$$= \frac{1}{\pi} \left[-\frac{4\pi^3}{n} - 0 \right] = -\frac{4\pi}{n}$$





Complex Variable Differentiation (2 Marks Questions)

4.1. Find $\lim_{z \rightarrow 0} \frac{\cos z - 1}{z}$.

Ans. $\lim_{z \rightarrow 0} \frac{\cos z - 1}{z} = \lim_{z \rightarrow 0} \frac{\cos z - \cos 0}{z - 0}$
 $= \cos'(0) = -\sin(0) = 0$

4.2. Write the Cauchy-Riemann equation for $f(z) = u + iv$ (a complex variable function).

Ans. The C-R equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

4.3. Write the Cauchy-Riemann equation in polar coordinates system.

AKTU 2015-16, 2017-18 (IV); Marks 02

Ans. Cauchy-Riemann equation in polar coordinate system :

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

and, $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$

4.4. Prove that the $f(z) = \sinh z$ is analytic.

AKTU 2016-17 (IV), Marks 02

Ans. Here $f(z) = u + iv = \sinh z = \sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$
 $u = \sinh x \cos y$ and $v = \cosh x \sin y$

$$\frac{\partial u}{\partial x} = \cosh x \cos y, \quad \frac{\partial u}{\partial y} = -\sinh x \sin y$$

$$\frac{\partial v}{\partial x} = \sinh x \sin y, \quad \frac{\partial v}{\partial y} = \cosh x \cos y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus C-R equations are satisfied.

Since $\sinh x$, $\cosh x$, $\sin y$ and $\cos y$ are continuous functions,

$\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also continuous functions satisfying C-R equations.

Hence $f(z)$ is analytic everywhere.

4.5. Prove that $u(x, y) = e^x \cos y$, is harmonic function.

AKTU 2017-18 (III), Marks 02

Ans.

$$u = e^x \cos y$$

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial^2 u}{\partial x^2} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y, \quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, hence u is a harmonic function.

4.6. Prove Cauchy-Riemann equation in polar form.

AKTU 2017-18 (III), Marks 02

Ans. Let (r, θ) be the polar coordinates of the point whose cartesian coordinates are (x, y) , then

$$x = r \cos \theta, y = r \sin \theta$$

$$z = x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

$$\therefore u + iv = f(z) = f(re^{i\theta}) \quad \dots(4.6.1)$$

Differentiating eq. (4.6.1) partially wrt r , we have

$$\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} = f'(re^{i\theta}) e^{i\theta}$$

Differentiating eq. (4.6.1) partially wrt θ , we have

$$\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} = f'(re^{i\theta}) ire^{i\theta} = ir \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right)$$

$$= -r \frac{\partial v}{\partial r} + ir \frac{\partial u}{\partial r}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r} \text{ and } \frac{\partial v}{\partial \theta} = r \frac{\partial u}{\partial r}$$

or $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$, which is the polar form of C-R equations.

4.7. Show that $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane.

AKTU 2021-22 (II), Marks 02

Ans. Since $f(z) = u + iv = z + 2\bar{z}$
then $f(x + iy) = x + iy + 2(x - iy) = 3x - iy$
 $u = 3x, v = -y$

$$\frac{\partial u}{\partial x} = 3, \frac{\partial v}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial y} = -1$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

So, Cauchy-Riemann condition is not satisfied. Thus $f(z)$ is not analytic anywhere in complex plane.

4.8. Find the image of $|z - 2i| = 2$ under the mapping $w = \frac{1}{z}$.

AKTU 2021-22 (II), Marks 02

Ans. Given : $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$
Here $z = x + iy$, and $w = u + iv$

$$z = \frac{1}{w} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$x + iy = \frac{u - iv}{u^2 + v^2}$$

Also, $x = \frac{u}{u^2 + v^2}, y = -\frac{v}{u^2 + v^2}$

Given $|z - 2i| = 2$

$$|x + iy - 2i| = 2$$

$$x^2 + (y - 2)^2 = 4$$

$$x^2 + y^2 - 4y + 4 = 4$$

$$x^2 + y^2 - 4y = 0$$

(Square both sides)

$$\left(\frac{u}{u^2 + v^2}\right) + \left(\frac{-v}{u^2 + v^2}\right)^2 = 4\left(\frac{-v}{u^2 + v^2}\right)$$

$$\Rightarrow \frac{u^2 + v^2}{(u^2 + v^2)^2} = 4\left(\frac{-v}{u^2 + v^2}\right)$$

$$1 = -4v \Rightarrow v = -\frac{1}{4}$$

Which is a straight line.

4.9. Show that complex function $f(z) = z^2$ is analytic.

AKTU 2018-19 (II), Marks 02

Ans.

$$f(z) = z^2 = (x + iy)^2 = x^2 - 3xy^2 + i(3x^2y - y^3)$$

Also

$$f(z) = u + iv$$

$$u = x^2 - 3xy^2, v = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 2x - 3y^2 \quad \dots(4.9.1)$$

$$\frac{\partial u}{\partial y} = -6xy \quad \dots(4.9.2)$$

$$\frac{\partial v}{\partial x} = 6xy \quad \dots(4.9.3)$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 \quad \dots(4.9.4)$$

Here, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence, $f(z)$ is an analytic function.

4.10. Define conformal mapping. AKTU 2018-19 (II), Marks 02

Ans. A mapping $w = f(z)$ is said to be conformal if the angle between any two smooth curves c_1, c_2 in the z -plane intersecting at the point z_0 is equal in magnitude and sense to the angle between their images c_1^*, c_2^* in the w -plane at the point $w_0 = f(z_0)$.

4.11. Write down the conditions for conformality.

Ans. A mapping $w = f(z)$ is conformal at each point z_0 , where $f(z)$ is analytic and $f'(z_0) \neq 0$.

4.12. Write down the properties of Mobius transformation.

Ans. Following are the properties of Mobius transformation :

1. Circles are transformed into circles under Mobius transformation.
2. The cross-ratio of four points is invariant under a bilinear transformation.





Complex Variable Integration (2 Marks Questions)

5.1. State Cauchy's integral theorem.

AKTU 2015-16 (III), Marks 02

Ans: If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C , then $\int_C f(z) dz = 0$

5.2. Write the statement of generalized Cauchy's integral formula for n^{th} derivative of an analytic function at the point $z = z_0$.

AKTU 2016-17 (IV), Marks 02

Ans: If a function $f(z)$ is analytic in a region, then its derivative at any point $z = z_0$ of that region is also analytic and is given by

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)} dz$$

Where, C is any closed contour in region surrounding the point $z = z_0$.

5.3. Evaluate $\int_C \frac{e^z}{z+1} dz$, where C is the circle $|z| = 2$.

AKTU 2016-17 (IV), Marks 02

Ans: The pole $z = -1$ lies inside the circle $|z| = 2$.
 $\therefore$ By Cauchy's integral formula,

$$\oint_C \frac{e^z}{z+1} dz = 2\pi i (e^z)_{z=-1} = \frac{2\pi i}{e}$$

5.4. Evaluate $\int_C \frac{z^2+1}{z^2-1} dz$, where C is circle $|z| = 3/2$.

AKTU 2016-17 (III), Marks 02

Ans:

$$I = \int_C \frac{z^2+1}{z^2-1} dz, |z| = 3/2$$

The integrand I is not analytic at the points $z = \pm 1$ and both lie inside C .

Now write the integrand as,

$$I = \int_C (z^2+1) \left[\frac{1}{2(z-1)} - \frac{1}{2(z+1)} \right] dz$$

$$I = \int_C \frac{z^2+1}{2(z-1)} dz - \int_C \frac{(z^2+1)}{2(z+1)} dz$$

$$I = I_1 + I_2$$

Now using Cauchy integral formula,

$$I_1 = \int_C \frac{(z^2+1)}{2(z-1)} dz = 2\pi i \left[\frac{(z-1)(z^2+1)}{2(z-1)} \right]_{z=1}$$

$$= 2\pi i \frac{(1+1)}{2} = 2\pi i$$

And

$$I_2 = \int_C \frac{-(z^2+1)}{2(z+1)} dz = -2\pi i \left[\frac{(z+1)(z^2+1)}{2(z+1)} \right]_{z=-1}$$

$$= -2\pi i \frac{(2)}{2} = -2\pi i$$

$$\therefore I = I_1 + I_2 = 2\pi i + (-2\pi i) = 0$$

5.5. Expand $f(z) = e^{z/(z-2)}$ in a Laurent series about the point $z = 2$.

AKTU 2021-22 (II), Marks 02

Ans: $f(z) = e^{z/(z-2)}$ at point $z = 2$
 $z - 2 = t$, so, $z = t + 2$
 $e^{z/(z-2)} = e^{(t+2)/t} = e^{1+2/t} = e \cdot e^{2/t}$

$$f(z) = e \cdot \left\{ 1 + \frac{2}{t(1!)} + \frac{(2)^2}{t^2(2!)} + \frac{(2)^3}{t^3(3!)} + \frac{(2)^4}{t^4(4!)} + \dots + \frac{(2)^n}{t^n(n!)} \right\}$$

5.6. Evaluate $\int_0^{1+i} (x^2 - iy) dz$ along the path $y = x$.

AKTU 2018-19 (II), Marks 02

Ans: Along line

$$\begin{aligned} y &= x \\ dy &= dx \\ dz &= dx + i dy = (1+i) dx \end{aligned}$$

$$\therefore \int_0^{1+i} (x^2 - iy) dz = \int_0^1 (x^2 - iy)(1+i) dx$$

$$= (1+i) \left[\frac{x^3}{3} - \frac{ix^2}{2} \right]_0^1$$

$$= (1+i)\left(\frac{1}{3} - \frac{1}{2}i\right) = \frac{5}{6} - \frac{1}{6}i$$

5.7. Find residue of $f(z) = \frac{\cos z}{z(z+5)}$ at $z=0$.

AKTU 2018-19 (II), Marks 02

Ans. $f(z) = \frac{\cos z}{z(z+5)}$

Poles of function are $z=0, z=-5$

Residue at $z=0$,

$$\begin{aligned} \text{Res } f(z) &= \lim_{z \rightarrow 0} z \times \frac{\cos z}{z(z+5)} \\ &= \frac{1}{5} \end{aligned}$$

5.8. Evaluate $\int_{|z|=1/2} \frac{e^z}{z^2+1} dz$

AKTU 2016-17 (III), Marks 02

Ans. Poles of integrand by putting denominator equal to zero

$$z^2 + 1 = 0$$

$$z = \pm i$$

The point $z = \pm i$ lie outside the circle $|z| = 1/2$.

∴ By Cauchy's integral theorem,

$$\int_{|z|=1/2} \frac{e^z}{z^2+1} dz = 0$$

5.9. Find the residue of $f(z) = \cot z$ at its pole.

AKTU 2016-17 (III), Marks 02

Ans. $f(z) = \cot z = \frac{\cos z}{\sin z}$

The poles of the function $f(z)$ are given by

$$\sin z = 0, z = n\pi, \text{ where } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

$$\text{Residue of } f(z) \text{ at } z = n\pi \text{ is } = \frac{\cos z}{\frac{d}{dz}(\sin z)}$$

$$\frac{\cos z}{\cos z} = 1$$

$$\left[\text{Residue at } (z=a) = \frac{\phi(a)}{\phi'(a)} \right]$$

5.10. Expand $\frac{1}{(z+1)(z+3)}$ in the regions $|z| < 1$.

AKTU 2015-16 (III), Marks 02

Ans. Let, $f(z) = \frac{1}{(z+1)(z+3)}$

Using partial fraction,

$$f(z) = \frac{1}{2} \left[\frac{1}{z+1} - \frac{1}{z+3} \right]$$

$$f(z) = \frac{1}{2} \left[\frac{1}{1+z} - \frac{1}{3+z} \right]$$

$$|z| < 1$$

$$f(z) = \frac{1}{2} \left[(1+z)^{-1} - \frac{1}{3} \left(1 + \frac{z}{3} \right)^{-1} \right]$$

$$f(z) = \frac{1}{2} [1 - z + z^2 - z^3 + \dots] - \frac{1}{6} \left[1 - \frac{z}{3} + \left(\frac{z}{3} \right)^2 - \left(\frac{z}{3} \right)^3 + \dots \right]$$

5.11. Discuss Singularity and its types.

AKTU 2017-18, (IV) Marks 02

Ans. Singularity: A singularity of a function $f(z)$ is a point at which the function ceases to be analytic.

Types of Singularities:

1. Isolated singularity,
2. Removable singularity,
3. Poles, and
4. Essential singularity.

5.12. Discuss singularity of $\frac{\cot \pi z}{(z-a)^2}$ at $z=a$ and $z=\infty$.

Ans. Let $f(z) = \frac{\cot \pi z}{(z-a)^2} = \frac{\cos \pi z}{\sin \pi z (z-a)^2}$

The poles are given by putting the denominator equal to zero.

$$\text{i.e., } \sin \pi z (z-a)^2 = 0 \Rightarrow (z-a)^2 = 0 \text{ or } \sin \pi z = 0 \Rightarrow \sin n\pi$$

$$z = a, \pi z = n\pi$$

$$z = a, n$$

$$(n \in \mathbb{I})$$

5.13. Discuss the nature of singularity of $\frac{\cot \pi z}{(z-a)^2}$ at $z=a$ and

$$z = \infty.$$

AKTU 2021-22 (II), Marks 02

Ans. Refer Q. 5.12, Page SQ-22F, Unit-5, 2 Marks Questions.

Nature of singularity: Non-isolated essential singularity.



B.Tech.
(SEM. II) EVEN SEMESTER THEORY
EXAMINATION, 2018-19
MATHEMATICS-II

Time : 3 Hours

Max. Marks : 100

Note : Attempt all Sections. If required any missing data; then choose suitably.

Section-A

1. Attempt all questions in brief : (2 × 10 = 20)

a. Find the PI of $\frac{d^2y}{dx^2} + 4y = \sin 2x$.

Refer Q. 1.4, Page SQ-2F, Unit-1, 2 Marks Questions.

b. Solve simultaneous equation $\frac{dx}{dt} = 3y, \frac{dy}{dt} = 3x$.

Refer Q. 1.6, Page SQ-3F, Unit-1, 2 Marks Questions.

c. Find the volume of solid generated by revolving the circle $x^2 + y^2 = 25$ about y-axis.

This question is out of syllabus from session (2022-23).

d. Evaluate $\left[-\frac{5}{2} \right]$, where Γ is gamma function

This question is out of syllabus from session (2022-23).

e. Find the Fourier constant a_1 of $f(x) = x^2, -\pi \leq x \leq \pi$.

Refer Q. 3.5, Page SQ-12F, Unit-3, 2 Marks Questions.

f. Discuss the convergence of sequence $a_n = \frac{2n}{n^2 + 1}$

Refer Q. 3.6, Page SQ-12F, Unit-3, 2 Marks Questions.

g. Show that complex function $f(z) = z^2$ is analytic.

Refer Q. 4.9, Page SQ-18F, Unit-4, 2 Marks Questions.

h. Define conformal mapping.

Refer Q. 4.10, Page SQ-18F, Unit-4, 2 Marks Questions.

i. Evaluate $\int_0^{1+i} (x^2 - iy)dz$ along the path $y = x$.

Refer Q. 5.6, Page SQ-20F, Unit-5, 2 Marks Questions.

j. Find residue of $f(z) = \frac{\cos z}{z(z+5)}$ at $z = 0$.

Refer Q. 5.7, Page SQ-21F, Unit-5, 2 Marks Questions.

Section-B

2. Attempt any three of the following : (10 × 3 = 30)

a. Use Frohenius method to solve

$$9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0.$$

This question is out of syllabus from session (2022-23).

b. Apply Dirichlet integral to find the volume of an octant of the sphere $x^2 + y^2 + z^2 = 25$.

This question is out of syllabus from session (2022-23).

c. Find half range sine series of $f(x) = \begin{cases} x & 0 < x < 2 \\ 4-x & 2 < x < 4 \end{cases}$

Refer Q. 3.25, Page 3-27F, Unit-3.

d. Show that $u = x^4 - 6x^2y^2 + y^4$ is harmonic function. Find complex function $f(z)$ whose u is a real part.

Refer Q. 4.23, Page 4-19F, Unit-4.

e. Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in regions

i. $1 < |z| < 2$

ii. $2 < |z|$

Refer Q. 5.26, Page 5-24F, Unit-5.

Section-C

3. Attempt any one parts of the following : (10 × 1 = 10)

a. Solve $\frac{d^2y}{dx^2} + y = \tan x$ by method of variation of parameters.

Refer Q. 1.23, Page 1-26F, Unit-1.

b. Solve $x^2 \frac{d^2y}{dx^2} - 2(x^2 + x) \frac{dy}{dx} + (x^2 + 2x + 2)y = 0$ by normal form.

This question is out of syllabus from session (2022-23).

4. Attempt any one parts of the following : (10 × 1 = 10)

- a. Prove that $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, where Γ is gamma function

Ans. This question is out of syllabus from session (2022-23).

- b. Use beta and gamma function to solve

i. $\int_0^\infty \frac{1}{1+x^4} dx$. ii. $\int_0^{\frac{\pi}{2}} \sqrt{\cot \theta} d\theta$.

Ans. This question is out of syllabus from session (2022-23).

5. Attempt any one parts of the following: (10 × 1 = 10)

- a. Find the Fourier series of $f(x) = x \sin x$ in $-\pi \leq x \leq \pi$.

Ans. Refer Q. 3.22, Page 3-24F, Unit-3.

- b. State D' Alembert's test. Test the series

$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \dots + \frac{x^n}{n^2 + 1} + \dots$$

Ans. Refer Q. 3.5, Page 3-5F, Unit-3.

6. Attempt any one parts of the following: (10 × 1 = 10)

- a. Let $f(z) = \frac{x^2 y^4 (x + iy)}{x^4 + y^{10}}$ when $z \neq 0$, $f(z) = 0$ when $z = 0$. Prove

that Cauchy-Riemann satisfies at $z = 0$ but function is not differentiable at $z = 0$.

Ans. Refer Q. 4.15, Page 4-10F, Unit-4.

- b. Find Mobius transformation that maps points $z = 0, -i, 2i$ into the points $w = 5i, \infty, -\frac{i}{3}$ respectively.

Ans. Refer Q. 4.35, Page 4-32F, Unit-4.

7. Attempt any one parts of the following: (10 × 1 = 10)

- a. Using Cauchy integral formula evaluate

$$\int \frac{\sin z}{(z^2 + 25)^2} dz, \text{ where } z \text{ is circle } |z| = 8.$$

Ans. Refer Q. 5.13, Page 5-12F, Unit-5.

- b. Apply residue theorem to evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)}$.

Ans. Refer Q. 5.36, Page 5-34F, Unit-5.



B. Tech.
(SEM. II) EVEN SEMESTER THEORY
EXAMINATION, 2021-22
ENGINEERING MATHEMATICS-II

Time : 3 Hours

Max. Marks : 100

Note : 1. Attempt all sections. If require any missing data; then choose suitably.

Section-A

- a. 1. Attempt all questions in brief. (2 × 10 = 20)

- a. Find the differential equation which represents the family of straight lines passing through the origins.

Ans. Refer Q. 1.11, Page SQ-5F, Unit-1, 2 Marks Questions.

- b. State the criterion for linearly independent solutions of the homogeneous linear n^{th} order differential equation.

Ans. Refer Q. 1.12, Page SQ-5F, Unit-1, 2 Marks Questions.

- c. Evaluate: $\int_0^1 \frac{dx}{\sqrt{-\log x}}$

Ans. This question is out of syllabus from session (2022-23).

- d. Find the volume of the solid obtained by rotating the ellipse $x^2 + 9y^2 = 9$ about the x -axis.

Ans. This question is out of syllabus from session (2022-23).

- e. Test the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$

Ans. Refer Q. 3.2, Page SQ-11F, Unit-3, 2 Marks Questions.

- f. Find the constant term when $f(x) = 1 + |x|$ is expanded in fourier series in the interval $(-3, 3)$.

Ans. Refer Q. 3.4, Page SQ-11F, Unit-3, 2 Marks Questions.

- g. Show that $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane.

Ans. Refer Q. 4.7, Page SQ-17F, Unit-4, 2 Marks Questions.

h. Find the image of $|z - 2i| = 2$ under the mapping $w = \frac{1}{z}$.

Ans. Refer Q. 4.8, Page SQ-17F, Unit-4, 2 Marks Questions.

i. Expand $f(z) = e^{2/(z-2)}$ in a Laurent series about the point $z = 2$.

Ans. Refer Q. 5.5, Page SQ-20F, Unit-5, 2 Marks Questions.

j. Discuss the nature of singularity of $\frac{\cot \pi z}{(z-a)^2}$ at $z = a$ and $z = \infty$.

Ans. Refer Q. 5.13, Page SQ-22F, Unit-5, 2 Marks Questions.

Section-B

2. Attempt any three of the following : (10 × 3 = 30)

a. Solve : $\frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x = e^{-t}$, $\frac{d^2y}{dt^2} - 4\frac{dx}{dt} + 3y = \sin 2t$.

Ans. Refer Q. 1.14, Page 1-16F, Unit-1.

b. Assuming $\lceil n \rceil (1 - n) = \pi \operatorname{cosec} n\pi$, $0 < n < 1$, show that

$$\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}; 0 < p < 1.$$

Ans. This question is out of syllabus from session (2022-23).

c. Test the series $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \dots$

Ans. Refer Q. 3.6, Page 3-6F, Unit-3.

d. If $f(z) = u + iv$ is an analytic function, find $f(z)$ in term of z if

$$u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x} \text{ when } f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}.$$

Ans. Refer Q. 4.26, Page 4-21F, Unit-4.

e. Evaluate by contour integration : $\int_0^{2\pi} e^{-\cos \theta} \cos(n\theta + \sin \theta) d\theta$; $n \in \mathbb{I}$.

Ans. Refer Q. 5.34, Page 5-32F, Unit-5.

Section-C

3. Attempt any one part of the following : (10 × 1 = 10)

a. Use the variation of parameter method to solve the differential equation $(D^2 - 1)y = 2(1 - e^{-2x})^{-1/2}$

Ans. Refer Q. 1.24, Page 1-26F, Unit-1.

b. Solve : $(1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4 \cos \log(1+x)$.

Ans. Refer Q. 1.18, Page 1-22F, Unit-1.

4. Attempt any one part of the following : (10 × 1 = 10)

a. The arc of the cardioid $r = a(1 + \cos \theta)$ included between $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ is rotated about the line $\theta = \frac{\pi}{2}$. Find the area of surface generated.

Ans. This question is out of syllabus from session (2022-23).

b. Evaluate $\iiint xyz \sin(x+y+z) dx dy dz$, the integral being extended to all positive values of the variables subject to the condition $x+y+z \leq \frac{\pi}{2}$.

Ans. This question is out of syllabus from session (2022-23).

5. Attempt any one part of the following : (10 × 1 = 10)

a. Test for convergence of the series $\frac{a+x}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + \dots$

Ans. Refer Q. 3.7, Page 3-7F, Unit-3.

b. Obtain fourier series for the function $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi < x < 0 \\ 1 - \frac{2x}{\pi}, & 0 < x < \pi \end{cases}$

$$\text{Hence deduce that } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

Ans. Refer Q. 3.20, Page 3-21F, Unit-3.

6. Attempt any one part of the following : (10 × 1 = 10)

a. Prove that $w = \frac{z}{1-z}$ maps the upper half of the z -plane onto upper half of the w -plane. What is the image of the circle $|z| = 1$ under this transformation ?

Ans. Refer Q. 4.33, Page 4-29F, Unit-4.

- b. Find a bilinear transformation which maps the point $i, -i, 1$ of the z - plane into $0, 1, \infty$ of the w - plane respectively.

Ans. Refer Q. 4.36, Page 4-32F, Unit-4.

7. Attempt any one part of the following : (10 × 1 = 10)

- a. Evaluate $\oint_C \frac{e^z}{z(1-z)^2} dz$, where C is

i. $|z| = \frac{1}{2}$

ii. $|z - 1| = \frac{1}{2}$

iii. $|z| = 2$.

Ans. Refer Q. 5.18, Page 5-17F, Unit-5.

- b. Find the Taylor's and Laurent's series which represent the

function $\frac{z^2 - 1}{(z + 2)(z + 3)}$ when

i. $|z| < 2$

ii. $2 < |z| < 3$

iii. $|z| > 3$.

Ans. Refer Q. 5.24, Page 5-23F, Unit-5.





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